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A RANKIN–SELBERG INTEGRAL FOR THE ADJOINT L-FUNCTION OF Sp_4

BY

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ABSTRACT

In this paper we study the Adjoint L-function for Sp_4. For generic cusp forms of $\mathrm{Sp}_4(\mathbb{A})$ we construct a global Rankin–Selberg integral which represents this L-function.

Introduction

In this paper we construct an integral representation for the partial adjoint L-function of Sp_4. More precisely, let π denote a *generic* cuspidal representation of $\mathrm{Sp}_4(\mathbb{A})$. Let Ad denote the ten dimensional irreducible adjoint representation of $\mathrm{SO}_5(\mathbb{C})$, the L-group of Sp_4. To this data one can associate the ten dimensional partial Adjoint L-function $L_S(\pi, \mathrm{Ad}, s)$. We shall construct a Rankin–Selberg integral which represents this L-function.

This construction uses an Eisenstein series on the double cover of Spin_9 and the Theta function on the double cover of Sp_4. The construction is in the spirit of the integral introduced in [G2], in the sense that it includes an integration over a unipotent subgroup which is of Heisenberg type. In the first section we introduce basic notations. In the second section we introduce the global Rankin–Selberg integral and show that it is Eulerian. Finally, in Section 3 we compute the local unramified integrals.

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1. Notation

1.1 Let G denote the group Spin_9. Since our construction involves a certain amount of computation we give several relations in G needed later. First we label the roots of G as

$$\begin{matrix} \alpha_1 & & \alpha_2 & & \alpha_3 & & \alpha_4 \\ 0 & - & 0 & - & 0 & = & 0\,. \end{matrix}$$

Given a root α we shall denote by $x_\alpha(r)$ the one dimensional unipotent subgroup corresponding to α. If $\alpha = \sum_{i=1}^4 n_i\alpha_i$ with $n_i \geq 0$ we shall write $(n_1n_2n_3n_4)$ to represent α.

A faithful representation for Spin_9 is obtained by restriction of the 16 dimensional representation of Spin_{10} to Spin_9. Using the embedding of Spin_{10} in E_6 we can easily give a matrix representation for Spin_9. We have

$$\begin{aligned} x_{1000}(r) &= I + r(e_{5,7} + e_{6,8} - e_{9,11} - e_{10,12}), \\ x_{0100}(r) &= I + r(e_{3,5} + e_{4,6} - e_{11,13} - e_{12,14}), \\ x_{0010}(r) &= I + r(e_{2,3} + e_{6,9} - e_{8,11} - e_{14,15}), \\ x_{0001}(r) &= I + r(e_{1,2} + e_{3,4} + e_{5,6} + e_{7,8} - e_{9,10} - e_{11,12} - e_{13,14} - e_{15,16})\,. \end{aligned}$$

Here I denotes the 16×16 identity matrix and $e_{i,j}$ is the 16×16 matrix which has 1 at the (i,j) position and zero elsewhere.

We shall denote the maximal torus of G as $h(t_1,t_2,t_3,t_4)$ where we parameterized it in a way that $h(t_1,1,1,1)$ denotes the maximal torus obtained from the embedding of the SL_2 corresponding to the simple root α_1, etc. Thus we can read from the Cartan matrix, the action of the torus on the roots of G. For the simple roots we have,

$$\begin{aligned} hx_{1000}(r)h^{-1} &= x_{1000}(t_1^2t_2^{-1}r), \\ hx_{0100}(r)h^{-1} &= x_{0100}(t_1^{-1}t_2^2t_3^{-1}r), \\ hx_{0010}(r)h^{-1} &= x_{0010}(t_2^{-1}t_3^2t_4^{-2}r), \\ hx_{0001}(r)h^{-1} &= x_{0001}(t_3^{-1}t_4^2r), \end{aligned}$$

where $h = h(t_1,t_2,t_3,t_4)$.

Let us remark that since $\mathrm{Spin}_9/\{\pm 1\} = \mathrm{SO}_9$ one can deduce most of the commutation relations and most of the matrix identities we need, from the standard matrix representation of SO_9.

Let W denote the Weyl group of G. We denote by w_i for $1 \leq i \leq 4$ the four simple reflections corresponding to the simple roots α_i. If $w = w_{i_1} w_{i_2} \cdots w_{i_k}$ we shall write $w = w(i_1 \cdots i_k)$ for short.

For our construction we need to consider two maximal parabolic subgroups of G. First let $Q = MU$ denote the maximal parabolic whose Levi part contains SL_4. Thus as an algebraic group M may be identified with $L = \{g \in \mathrm{GL}_4 : \det g$ is a square$\}$. U is a two step unipotent group which consists of all positive roots $\alpha = \sum n_i \alpha_i$ with $n_4 \geq 1$. Let us describe more explicitly the identification of M with L. The simple roots are identified as

$$x_{\alpha_1}(r) \to \begin{pmatrix} 1 & r & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix},$$

$$x_{\alpha_2}(r) \to \begin{pmatrix} 1 & & & \\ & 1 & r & \\ & & 1 & \\ & & & 1 \end{pmatrix},$$

$$x_{\alpha_3}(r) \to \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & r \\ & & & 1 \end{pmatrix}.$$

Thus from the action of the torus on the simple roots we have the following identification:

$$h(t_1, t_2, t_3, t_4) \to \begin{pmatrix} t_1 & & & \\ & t_1^{-1} t_2 & & \\ & & t_2^{-1} t_3 & \\ & & & t_3^{-1} t_4^2 \end{pmatrix}.$$

The action of W on $h(t_1, t_2, t_3, t_4)$ can be read via this embedding. Namely, the action of w_i for $1 \leq i \leq 3$ is the obvious one, and w_4 acts as

$$w_4 \cdot \mathrm{diag}(a_1, a_2, a_3, a_4) = \mathrm{diag}(a_1, a_2, a_3, a_4^{-1})$$

where $a_1 a_2 a_3 a_4$ is a square.

Let $R = \mathrm{Sp}_4 U \subset MU = Q$ where

$$\mathrm{Sp}_4 = \left\{ g \in \mathrm{GL}_4 : g^t \begin{pmatrix} & & & 1 \\ & & 1 & \\ & -1 & & \\ -1 & & & \end{pmatrix} g = \begin{pmatrix} & & & 1 \\ & & 1 & \\ & -1 & & \\ -1 & & & \end{pmatrix} \right\}$$

and Sp_4 is embedded in M in the obvious way, i.e. via the identification of M with L.

The second maximal parabolic is $P = \mathrm{GL}_3 \cdot \mathrm{Spin}_3 V$. Thus V consists of all roots $\alpha = \sum n_i \alpha_i$ for which $n_3 \geq 1$.

We will need to study the space $P \backslash G/R$. We have

LEMMA 1.1: *There are six double cosets in $P\backslash G/R$. As representatives we can choose e; $w(34)$; $w(3214)$; $w(32434)$; $w(3423421)$ and $w(342341234)$.*

Proof: First consider $P\backslash G/Q$. It is not difficult to check that $|P\backslash G/Q| = 4$ and as representatives we may choose e; $w(34)$; $w(32434)$ and $w(342341234)$. As representatives for $P\backslash G/R$ we may choose the set wv_w where $w \in P\backslash G/Q$ as above and $v_w \in (w^{-1}Pw \cap \mathrm{SL}_4)\backslash \mathrm{SL}_4 / \mathrm{Sp}_4$. If $w = e$ or $w = w(342341234)$ then $w^{-1}Pw \cap \mathrm{SL}_4$ is a parabolic subgroup of SL_4 whose Levi part is GL_3. Thus $v_w = e$ in these cases. In the case of $w = w(34)$ or $w = w(32434)$ the group $w^{-1}Pw \cap \mathrm{SL}_4$ contains a parabolic subgroup of SL_4 whose Levi part is $\mathrm{GL}_1 \times \mathrm{GL}_2$. In this case $|w^{-1}Pw \cap \mathrm{SL}_4 \backslash \mathrm{SL}_4 / \mathrm{Sp}_4| = 2$ and as representatives we may choose $v_w = e$ and $v_w = w(21)$. Thus the lemma follows. ∎

1.2 For the construction of our Eisenstein series we need to consider the double cover of $G = \mathrm{Spin}_9$. In [M], Matsumoto constructed a unique double cover for the group G. We shall denote this group $\tilde{G}$. It follows from [M] that there is a cocycle σ of $G \times G$ such that

$$\sigma\big(h(t_1,t_2,t_3,t_4), h(r_1,r_2,r_3,r_4)\big) = (t_1, r_1 r_2)(t_2, r_2 r_3)(t_3, r_3)$$

where $(\ ,\ )$ is the two order Hilbert symbol.

Thus we shall identify $\tilde{G}$ with the group of all pairs $\langle g, \varepsilon\rangle$ with $g \in G$ and $\varepsilon \in \{\pm 1\}$ so that $\langle g_1, \varepsilon_1\rangle\langle g_2, \varepsilon_2\rangle = \langle g_1 g_2, \sigma(g_1, g_2)\varepsilon_1\varepsilon_2\rangle$.

Given a subgroup H of G, $\tilde{H}$ will denote its full inverse image in $\tilde{G}$. If there is a splitting homomorphism for H we shall identify H with its homomorphic image in $\tilde{G}$. When needed we shall describe the homomorphism explicity. When there is no confusion we shall write h for $\langle h, 1\rangle$.

It is not difficult to check that when restricting from $\tilde{G}$ to $\tilde{M}$ — the double cover of M — one obtains on M, via its identification with L, the cocycle described in [K-P]. By abuse of notations we shall denote by σ the cocycle of $\widetilde{\mathrm{GL}}_4$ whose restriction to $\tilde{L}$ coincides with the restriction from $\tilde{G}$. Also, since $\mathrm{Spin}_3 \subset P$ splits under the cover it follows that $P_0 = \widetilde{\mathrm{GL}}_3 \cdot \mathrm{Spin}_3 \cdot V$ is a well defined subgroup of $\tilde{P}$. The restriction of σ to $\widetilde{\mathrm{GL}}_3$ is as given in [K-P], page 41 with $c = 1$.

1.3. In this section we review some of the properties of the Weil representation. We refer the reader to [M-V-W] for details.

Let H_5 denote the Heisenberg group with five variables. We shall identify H_5 with the group of all (x_1, x_2, y_1, y_2, z) with product given by

$$(x_1, x_2, y_1, y_2, z)(x_1', x_2', y_1', y_2', z') =$$
$$(x_1 + x_1', x_2 + x_2', y_1 + y_1', y_2 + y_2', z + z' + x_1 y_2' + x_2 y_1' - y_1 x_2' - y_2 x_1').$$

We shall also write (x, y, z) as elements of H_5 where $x = (x_1, x_2)$ and $y = (y_1, y_2)$.

Let F be a global field and $\mathbf{A}$ its ring of adeles. Let ψ denote a nontrivial additive character of $F\backslash\mathbf{A}$. Let $\mathcal{S}(\mathbf{A}^2)$ denote the space of all Schwartz functions on $\mathbf{A}^2$. The Weil representation ω_ψ is a representation of $H_5(\mathbf{A})\widetilde{\mathrm{Sp}}_4(\mathbf{A})$ which acts on $\mathcal{S}(\mathbf{A}^2)$. Here we define $\widetilde{\mathrm{Sp}}_4$ as follows. Let Sp_4 be as defined in Section 1.1. We define $\widetilde{\mathrm{Sp}}_4$ as the group of all $\langle h, \varepsilon\rangle_1$ where $h \in \mathrm{Sp}_4$ and $\varepsilon \in \{\pm 1\}$ with product $\langle h_1, \varepsilon_1\rangle_1 \langle h_2, \varepsilon_2\rangle_1 = \langle h_1 h_2, \sigma_1(h_1, h_2)\varepsilon_1\varepsilon_2\rangle_1$ where $\sigma_1(h_1, h_2)$ is the cocycle obtained by restricting the cocycle of $\widetilde{\mathrm{GL}}_4$ as defined in [K-P]. Thus

$$\sigma_1\left(\begin{pmatrix} a_1 & & & \\ & a_2 & & \\ & & a_2^{-1} & \\ & & & a_1^{-1} \end{pmatrix}, \begin{pmatrix} b_1 & & & \\ & b_2 & & \\ & & b_2^{-1} & \\ & & & b_1^{-1} \end{pmatrix}\right) = (a_1, b_1)(a_2, b_2)\ .$$

Returning to the Weil representation, its action is given by

$$\omega_\psi\left[(0, y, z)(x, 0, 0)\right]\phi(\xi) = \phi(\xi + x)\psi\left(\xi \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} y^t + z\right),$$

$$\omega_\psi\left(\left\langle \begin{pmatrix} a & & & \\ & b & & \\ & & b^{-1} & \\ & & & a^{-1} \end{pmatrix}, \varepsilon \right\rangle\right) \phi(\xi_1, \xi_2) = \varepsilon\gamma_{ab}(a, b)|ab|^{1/2}\phi(a\xi_1, b\xi_2),$$

$$\omega_\psi\left(\left\langle \begin{pmatrix} 1 & r & & \\ & 1 & & \\ & & 1 & -r \\ & & & 1 \end{pmatrix}, \varepsilon \right\rangle\right) \phi(\xi) = \varepsilon\phi\left(\xi \begin{pmatrix} 1 & r \\ & 1 \end{pmatrix}\right),$$

$$\omega_\psi\left(\left\langle \begin{pmatrix} 1 & & r_1 & r_2 \\ & 1 & r_3 & r_1 \\ & & 1 & \\ & & & 1 \end{pmatrix}, \varepsilon \right\rangle\right) \phi(\xi) = \varepsilon\psi\left(1/2\xi \begin{pmatrix} r_1 & r_2 \\ r_3 & r_1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \xi^t\right) \phi(\xi)\ .$$

Here $\xi = (\xi_1, \xi_2)$, $a, b \in \mathbf{A}^*$, $\varepsilon \in \{\pm 1\}$, $r_i \in F$, $\phi \in \mathcal{S}(\mathbf{A}^2)$, and γ_t denotes the Weil constant (see [W]).

Next we define the theta function on the group $H_5(\mathbf{A}) \cdot \widetilde{\mathrm{Sp}}_4(\mathbf{A})$ by

$$\tilde{\theta}_\phi(hg) = \sum_{\xi \in F^2} \omega_\psi(hg)\phi(\xi)$$

where $h \in H_5(\mathbf{A})$, $g \in \widetilde{\mathrm{Sp}}_4(\mathbf{A})$ and $\phi \in \mathcal{S}(\mathbf{A}^2)$.

Finally we define a homomorphism $\tau\colon U \to H_5$. For $u \in U$ write

$$u = x_{0001}(x_1)x_{0011}(x_2)x_{0111}(y_1)x_{1111}(y_2)x_{1112}(r_1)x_{0122}(r_2)u'$$

where u' is a product of all other one dimensional unipotent subgroups in U in any fixed order. Thus the roots in u' include (0012); (0112); (1122) and (1222). We define

$$\tau(u) = (x_1, x_2, y_1, y_2, r_1 + r_2)\ .$$

This is a well defined homomorphism from U onto H_5.

1.4 Let π be a cuspidal representation of $\mathrm{Sp}_4(\mathbf{A})$. As usual we shall realize π in the space of $L^2_{\mathrm{cusp}}\big(\mathrm{Sp}_4(F)\backslash \mathrm{Sp}_4(\mathbf{A})\big)$. We will assume that π is generic. This means that there exists $\varphi \in \pi$ and $\alpha, \beta \in F^*$ such that the function

$$W_\varphi^{(\alpha,\beta)}(g) = \int_{(F\backslash\mathbf{A})^4} \varphi\left[\begin{pmatrix} 1 & r_1 & & \\ & 1 & & \\ & & 1 & -r_1 \\ & & & 1 \end{pmatrix}\begin{pmatrix} 1 & & r_3 & r_4 \\ & 1 & r_2 & r_3 \\ & & 1 & \\ & & & 1 \end{pmatrix} g\right] \psi(\alpha r_1 + \beta r_2)dr_i$$

is nonzero for some $g \in \mathrm{Sp}_4(\mathbf{A})$. Write $\beta = \lambda\mu^2$ where λ is square free. Since

$$W_\varphi^{(\alpha,\beta)}(g) = W_\varphi^{(1,\lambda)}\left[\begin{pmatrix} \alpha & & & \\ & \mu & & \\ & & \mu^{-1} & \\ & & & \alpha^{-1} \end{pmatrix} g\right]$$

it is enough to consider the case where $\alpha = 1$ and β is square free.

1.5. In this section we construct the Eisenstein series we use. Let $\tilde{\theta}$ denote the theta function on $\widetilde{\mathrm{GL}}_3(\mathbf{A})$ — the double cover of $\mathrm{GL}_3(\mathbf{A})$. This function is constructed in [K-P] and for the properties we need see also [B-G]. Let γ_t for $t \in \mathbf{A}^*$ denote the global Weil constant. We extend $\gamma \circ \det$ to a function of $\widetilde{\mathrm{GL}}_3(\mathbf{A})$ in the obvious way. Next we extend the representation $(\gamma \circ \det)\tilde{\theta}$ from

$\widetilde{\mathrm{GL}}_3(\mathbb{A})$ to $P_0(\mathbb{A})$ by letting it act trivially on $\mathrm{Spin}_3(\mathbb{A})V(\mathbb{A})$. Let δ_P denote the modular function of P. We view it as a function of P_0 by composing it with the projection $P_0 \to P$. Given $s \in \mathbb{C}$ let

$$I(s) = \mathrm{Ind}_{P_0(\mathbb{A})}^{\tilde{G}(\mathbb{A})} (\gamma \circ \det)\tilde{\theta} \otimes \delta_P^s .$$

Thus $F_s \in I(s)$ is a smooth function $F_s : \tilde{G}(\mathbb{A}) \to \tilde{\theta}$ satisfying

$$F_s(p_0 g) = \delta_P^s(p_0)\gamma(\det m)\tilde{\theta}(m)F_s(g)$$

for all $p_0 = mr$ where $m \in \widetilde{\mathrm{GL}}_3(\mathbb{A})$ and $r \in \mathrm{Spin}_3(\mathbb{A})V(\mathbb{A})$ and for all $g \in \tilde{G}(\mathbb{A})$. Here $\tilde{\theta}(m)F_s(g)$ denotes the action of the theta representation.

Since $\tilde{\theta}$ is automorphic (see [K-P]) there is a $\mathrm{GL}_3(F)$ invariant form $\ell\colon \tilde{\theta} \to \mathbb{C}$. Set $\tilde{f}_s(g) = \ell(F_s(g))$. Thus we may define the Eisenstein series

$$\tilde{E}(g, \tilde{f}_s, s) = \sum_{\gamma \in P(F)\backslash G(F)} \tilde{f}_s(\gamma g).$$

This series converges for $\mathrm{Re}(s)$ large and admits a meromorphic continuation to the whole complex plane.

2. The global integral

Let φ be a generic cusp form on $\mathrm{Sp}_4(\mathbb{A})$. By Section 1.4 we may assume that $W_\varphi^{(1,\lambda)}(g) \not\equiv 0$ for some λ. Since we may choose any nontrivial additive character for the construction of the Weil representation, one can check that our construction of the global integral, which we shall soon define, is valid for any choice of $\lambda \in F^*$. Hence we shall assume that π is such that

$$W_\varphi(g) = \int_{F\backslash\mathbb{A}} \varphi\left[\begin{pmatrix} 1 & r_1 & & \\ & 1 & & \\ & & 1 & -r_1 \\ & & & 1 \end{pmatrix}\begin{pmatrix} 1 & & r_3 & r_4 \\ & 1 & r_2 & r_3 \\ & & 1 & \\ & & & 1 \end{pmatrix} g\right] \psi(-r_1 + 1/2r_2)dr_i$$

is nonzero for some φ and $g \in \mathrm{Sp}_4(\mathbb{A})$.

Let $\mathrm{Sp}_4(\mathbb{A})$ be embedded in $G(\mathbb{A})$ via its embedding in $L(\mathbb{A}) \subset Q(\mathbb{A})$. We denote this embedding by j. We define

(2.1)
$$I(\varphi, \phi, \tilde{f}_s, s) = \int_{\mathrm{Sp}_4(F)\backslash \mathrm{Sp}_4(\mathbb{A})} \int_{U(F)\backslash U(\mathbb{A})} \varphi(g)\tilde{\theta}_\phi(\tau(u)g)\tilde{E}(uj(g), \tilde{f}_s, s)du dg.$$

Here $j(g)$ (resp. g) stands for $< j(g), 1 >$ (resp. $< g, 1 >_1$). Thus the integral is well defined. Note also that if we define a character $\tilde{\psi}$ on $[U, U]$ as

$$\tilde{\psi}(x_{1112}(r_1)x_{0122}(r_2)u') = \psi(r_1 + r_2)$$

where u' is a product of all other roots in $[U, U]$ in any fixed order, then Sp_4 fixes $\tilde{\psi}$ under the obvious action of Sp_4 on $[U, U]$. Thus the U integration is indeed invariant under $\mathrm{Sp}_4(F)$. Since φ is a cusp form, (2.1) converges absolutely for all s for which the Eisenstein series has no poles.

To shorten the notation, we shall write g for $j(g)$. Given $\tilde{f}_s(g)$ as in Section 1.5 let

$$\tilde{f}_W(g, s) = \int_{(F\backslash \mathbb{A})^3} \tilde{f}_s(x_{1000}(r_1)x_{0100}(r_2)x_{1100}(r_3)g)\psi(r_1)dr_i \ .$$

Notice that the above 3 roots consist of the maximal unipotent subgroup of GL_3 as embedded in P. Thus

$$\tilde{f}_W \in \mathrm{Ind}_{P_0(\mathbb{A})}^{\tilde{G}(\mathbb{A})} \delta_P^s \otimes (\gamma \circ \det)\mathcal{W}(\tilde{\theta}, \psi)$$

where $\mathcal{W}(\tilde{\theta}, \psi)$ is the space of all functions of the form

$$h \to \int \tilde{\theta}\left[\begin{pmatrix} 1 & r_1 & r_3 \\ & 1 & r_2 \\ & & 1 \end{pmatrix} h\right] \psi(r_1)dr_i,$$

$h \in \widetilde{\mathrm{GL}}_3(\mathbb{A})$. It follows from [P-P.S] and [B-G] that $\mathcal{W}(\tilde{\theta}, \psi) \not\equiv 0$ and that it is factorizable.

Let $U_0 \subset U$ be the subgroup consisting of all roots in U omitting the root $x_{1111}(r)$. Thus $\dim U_0 = 9$. Let N denote the maximal unipotent subgroup of Sp_4 consisting of upper triangular matrices. Finally, set $w_0 = w(342341234)$. We have

THEOREM 2.1: *For* $\mathrm{Re}(s)$ *large,*

$$I(\varphi, \phi, \tilde{f}_s, s) = \int_{N(\mathbb{A})\backslash \mathrm{Sp}_4(\mathbb{A})} \int_{U_0(\mathbb{A})} W_\varphi(g)\omega_\psi(\tau(u)g)\phi(0, 1)\tilde{f}_W(w_0ug, s)dudg \ .$$

Proof: For simplicity we shall write I for $I(\varphi, \phi, \tilde{f}_s, s)$. For $\mathrm{Re}(s)$ large we unfold the Eisenstein series, and using Lemma 1.1 we get

$$I = \sum_w \int_{R^w(F)\backslash \mathrm{Sp}_4(\mathbb{A})U(\mathbb{A})} \varphi(g)\tilde{\theta}_\phi(\tau(u)g)\tilde{f}_s(wug)dudg$$

where w runs over the representatives of $P\backslash G/R$ as described in Lemma 1.1 and $R^w = w^{-1}Pw \cap R$. First we claim that if w is such that $wx_{1112}(r)w^{-1} \in V$ or $wx_{0122}(r_2)w^{-1} \in V$ then the contribution of this w to I is zero. Indeed, this follows from the fact that

$$\tilde{\theta}_\phi\Big(\tau(u)\tau\big(x_{1112}(r_1)x_{0112}(r_2)\big)g\Big) = \psi(r_1 + r_2)\tilde{\theta}_\phi\big(\tau(u)g\big)$$

and hence we shall end up integrating ψ on $F\backslash\mathbb{A}$.

If $w = e$ then clearly $wx_{1112}(r_1)w^{-1} \in V$. For $w = w(34)$ we have $wx_{0122}w^{-1} = x_{0112} \in V$. For $w = w(3214)$ we have $wx_{0122}w^{-1} = x_{1222} \in V$ and for $w = w(3423421)$ we have $wx_{0122}w^{-1} = x_{1112} \in V$.

Next consider $w = w(32434)$. We have

$$\begin{array}{lll} wx_{0100}w^{-1} = x_{0122}; & wx_{1100}w^{-1} = x_{1122}; & wx_{0110}w^{-1} = x_{0112}; \\ wx_{1110}w^{-1} = x_{1222}; & wx_{0111}w^{-1} = x_{0001}; & wx_{1111}w^{-1} = x_{1111}. \end{array}$$

Before we proceed let us express the embedding of Sp_4 in terms of the roots in G. Recall that Sp_4 is embedded in G via the embedding in SL_4 which is a subgroup of L. The simple positive roots of SL_4, as embedded in G, are (1000); (0100) and (0010). From this it is easy to deduce that N, the maximal unipotent subgroup of Sp_4, is expressed in terms of the positive roots of G as follows:

$$\begin{pmatrix} 1 & r & & \\ & 1 & & \\ & & 1 & -r \\ & & & 1 \end{pmatrix} \to x_{1000}(r)x_{0010}(-r),$$

$$\begin{pmatrix} 1 & & r & \\ & 1 & & r \\ & & 1 & \\ & & & 1 \end{pmatrix} \to x_{1100}(r)x_{0110}(r),$$

$$\begin{pmatrix} 1 & & & \\ & 1 & r & \\ & & 1 & \\ & & & 1 \end{pmatrix} \to x_{0100}(r),$$

$$\begin{pmatrix} 1 & & & r \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix} \to x_{1110}(r)\ .$$

Returning back to $w = w(32434)$ we see from the way the roots are permuted by w that R^w contains as a normal subgroup the group generated by the roots

$x_{0100}(r); x_{1100}(r)x_{0110}(r); x_{1110}(r)$; $x_{0111}(r)$ and $x_{1111}(r)$. Notice that the first three form the Siegel radical in Sp_4. Notice also that $\tilde{f}_s(wx_\alpha(r)g,s) = \tilde{f}_s(wg,s)$ for $r \in \mathbf{A}$ and α is any of the above roots. Thus after a suitable change of variables we obtain as an inner integral

$$(2.2)\quad \int_{(F\backslash\mathbf{A})^5} \varphi\left[\begin{pmatrix}1 & & r_3 & r_4\\ & 1 & r_2 & r_3\\ & & 1 & \\ & & & 1\end{pmatrix} g\right] \tilde{\theta}_\phi\left[(0,y,0)\begin{pmatrix}1 & & r_3 & r_4\\ & 1 & r_2 & r_3\\ & & 1 & \\ & & & 1\end{pmatrix} g\right] dr_i dy$$

where as before $y = (y_1, y_2)$. We have (see Section 1.3)

$$\tilde{\theta}_\phi\left[(0,y,0)\begin{pmatrix}1 & & r_3 & r_4\\ & 1 & r_2 & r_3\\ & & 1 & \\ & & & 1\end{pmatrix} g\right]$$
$$= \sum_{\xi\in F^2} \omega_\psi\left[\begin{pmatrix}1 & & r_3 & r_4\\ & 1 & r_2 & r_3\\ & & 1 & \\ & & & 1\end{pmatrix} g\right] \phi(\xi)\psi(\xi_1 y_1 + \xi_2 y_2)$$

where $\xi = (\xi_1, \xi_2)$. Thus carrying out the y integration in (2.2) we obtain

$$\int_{(F\backslash\mathbf{A})^3} \varphi\left[\begin{pmatrix}1 & & r_3 & r_4\\ & 1 & r_2 & r_3\\ & & 1 & \\ & & & 1\end{pmatrix} g\right] \omega_\psi\left[\begin{pmatrix}1 & & r_3 & r_4\\ & 1 & r_2 & r_3\\ & & 1 & \\ & & & 1\end{pmatrix} g\right] \phi(0,0) dr_i$$
$$= \omega_\psi(g)\phi(0) \int_{(F\backslash\mathbf{A})^3} \varphi\left[\begin{pmatrix}1 & & r_3 & r_4\\ & 1 & r_2 & r_3\\ & & 1 & \\ & & & 1\end{pmatrix} g\right] dr_i\ .$$

By cuspidality of φ the last integral vanishes. Thus the contribution from w to I is zero. We are left with $w = w_0$. To compute R^{w_0} we use the following relations:

$$(2.3)\quad \begin{array}{ll} w_0 x_{0100} w_0^{-1} = x_{0100}; & w_0 x_{1000} w_0^{-1} = x_{0012};\\ w_0 x_{0010} w_0^{-1} = x_{1000}; & w_0 x_{1100} w_0^{-1} = x_{0112};\\ w_0 x_{0110} w_0^{-1} = x_{1122}; & w_0 x_{1110} w_0^{-1} = x_{1112};\\ w_0 x_{1111} w_0^{-1} = x_{0001}. & \end{array}$$

From this we may conclude that $R^{w_0} = Q_1\langle x_{1111}(y_2)\rangle$ where Q_1 is the parabolic subgroup of Sp_4 which preserves a line. Thus

$$I = \int_{Q_1(F)\backslash \mathrm{Sp}_4(\mathbf{A})} \int_{F\backslash\mathbf{A}} \int_{U_0(\mathbf{A})} \varphi(g)\tilde{\theta}_\phi((0,0,0,y_2,0)r(u)g)\tilde{f}_s(w_0 ug) dy_2 du dg\ .$$

Using Section 1.3 we have

$$\int_{F\backslash\mathbb{A}} \tilde{\theta}_\phi\big((0,0,0,y_2,0)\tau(u)g\big)dy_2 = \sum_{\xi_1,\xi_2\in F} \omega_\psi\big(\tau(u)g\big)\phi(\xi_1,\xi_2)\int_{F\backslash\mathbb{A}}\psi(\xi_1 y_2)$$
$$= \sum_{\xi\in F}\omega_\psi\big(\tau(u)g\big)\phi(0,\xi)\ .$$

Thus

$$I = \int_{Q_1(F)\backslash \mathrm{Sp}_4(\mathbb{A})}\int_{U_0(\mathbb{A})}\varphi(g)\sum_{\xi\in F}\omega_\psi\big(\tau(u)g\big)\phi(0,\xi)\tilde{f}_s(w_0ug)dudg\ .$$

Write $Q_1 = (\mathrm{GL}_1\times\mathrm{SL}_2)\cdot N_1$. Thus

$$N_1 = \left\{\begin{pmatrix}1 & z_1 & & \\ & 1 & & \\ & & 1 & -z_1\\ & & & 1\end{pmatrix}\begin{pmatrix}1 & & z_2 & z_3\\ & 1 & & z_2\\ & & 1 & \\ & & & 1\end{pmatrix}\right\}$$

and it is embedded in G as

$$x_{1000}(z_1)x_{0010}(-z_1)x_{0110}(z_2)x_{1100}(z_2)x_{1110}(z_3)\ .$$

From relations (2.3) we obtain

$$\tilde{f}_s\big(w_0x_{1000}(z_1)x_{0010}(-z_1)x_{0110}(z_2)x_{1100}(z_2)x_{1110}(z_3)ug\big) = \tilde{f}_s\big(x_{1000}(z_1)x_{1100}(z_2)w_0ug\big). \tag{2.4}$$

Also from Section 1.3 we deduce

$$\omega_\psi\left[\begin{pmatrix}1 & z_1 & & \\ & 1 & & \\ & & 1 & -z_1\\ & & & 1\end{pmatrix}\begin{pmatrix}1 & & z_2 & z_3\\ & 1 & & z_2\\ & & 1 & \\ & & & 1\end{pmatrix}\tau(u)g\right]\phi(0,\xi) = \omega_\psi\big(\tau(u)g\big)\phi(0,\xi)\ .$$

From this and from (2.4) we get

$$I = \int_{\mathrm{GL}_1(F)\,\mathrm{SL}_2(F)N_1(\mathbb{A})\backslash\,\mathrm{Sp}_4(\mathbb{A})}\int_{(F\backslash\mathbb{A})^3}\int_{U_0(\mathbb{A})}$$
$$\varphi\left[\begin{pmatrix}1 & z_1 & & \\ & 1 & & \\ & & 1 & -z_1\\ & & & 1\end{pmatrix}\begin{pmatrix}1 & & z_2 & z_3\\ & 1 & & z_2\\ & & 1 & \\ & & & 1\end{pmatrix}g\right]$$
$$\sum_{\xi\in F}\omega_\psi\big(\tau(u)g\big)\phi(0,\xi)\tilde{f}_s\big(x_{1000}(z_1)x_{1100}(z_2)w_0ug\big)dz_idudg\ .$$

Next we consider the following Fourier expansion:

$$\int_{F\backslash\mathbb{A}}\varphi\left[\begin{pmatrix}1&&&z_3\\&1&&\\&&1&\\&&&1\end{pmatrix}g\right]dz_3$$

$$=\sum_{\eta_1,\eta_2\in F}\int_{(F\backslash\mathbb{A})^3}\varphi\left[\begin{pmatrix}1&r_1&r_2&z_3\\&1&0&r_2\\&&1&-r_1\\&&&1\end{pmatrix}g\right]\psi(\eta_1r_1+\eta_2r_2)dr_idz_3.$$

The group $\mathrm{GL}_1(F)\times\mathrm{SL}_2(F)$, as embedded above, acts on the set (η_1,η_2) with two orbits. By cuspidality of φ, the trivial orbit contributes zero to the above integral. For the open orbit representative we choose (-1,0). The stabilizer in $\mathrm{GL}_1(F)\times\mathrm{SL}_2(F)$ is $\mathrm{GL}_1^{\Delta}(F)\cdot N_2(F)$ embedded in $\mathrm{Sp}_4(F)$ as

$$\begin{pmatrix}\alpha&&&\\&\alpha&&\\&&\alpha^{-1}&\\&&&\alpha^{-1}\end{pmatrix}\begin{pmatrix}1&&&\\&1&r&\\&&1&\\&&&1\end{pmatrix}.$$

Thus after a suitable change of variables in r_i we obtain

$$I=\int_{\mathrm{GL}_1^{\Delta}(F)N_2(F)N_1(\mathbb{A})\backslash\mathrm{Sp}_4(\mathbb{A})}\int_{(F\backslash\mathbb{A})^5}\int_{U_0(\mathbb{A})}$$
$$\varphi\left[\begin{pmatrix}1&r_1&&\\&1&&\\&&1&-r_1\\&&&1\end{pmatrix}\begin{pmatrix}1&&r_2&z_3\\&1&&r_2\\&&1&\\&&&1\end{pmatrix}g\right]$$
$$\sum_{\xi\in F}\omega_\psi(\tau(u)g)\phi(0,\xi)\tilde f_s(x_{1000}(z_1)x_{1100}(z_2)w_0ug)\psi(-r_1+z_1)dr_idz_jdudg\ .$$

Thus, since $N=N_1N_2$,

$$I=\int_{\mathrm{GL}_1^{\Delta}(F)N(\mathbb{A})\backslash\mathrm{Sp}_4(\mathbb{A})}\int_{(F\backslash\mathbb{A})^6}\int_{U_0(\mathbb{A})}$$
$$\varphi\left[\begin{pmatrix}1&r_1&&\\&1&&\\&&1&-r_1\\&&&1\end{pmatrix}\begin{pmatrix}1&&r_2&z_3\\&1&m&r_2\\&&1&\\&&&1\end{pmatrix}g\right]$$
$$\sum_{\xi\in F}\omega_\psi\left[\begin{pmatrix}1&&&\\&1&m&\\&&1&\\&&&1\end{pmatrix}\tau(u)g\right]\phi(0,\xi)$$
$$\tilde f_s\left[x_{1000}(z_1)x_{0100}(m)x_{1100}(z_2)w_0ug\right]\psi(-r_1+z_1)dmdr_idz_jdudg\ .$$

Consider the contribution from $\xi = 0$. By Section 1.3

$$\omega_\psi \left[\begin{pmatrix} 1 & & & \\ & 1 & m & \\ & & 1 & \\ & & & 1 \end{pmatrix} \tau(u)g \right] \phi(0,0) = \omega_\psi\big(\tau(u)g\big).$$

Thus we obtain as an inner integral

$$\int_{(F\backslash\mathbb{A})^6} \varphi \left[\begin{pmatrix} 1 & r_1 & & \\ & 1 & & \\ & & 1 & -r_1 \\ & & & 1 \end{pmatrix} \begin{pmatrix} 1 & & r_2 & z_3 \\ & 1 & m & r_2 \\ & & 1 & \\ & & & 1 \end{pmatrix} g \right]$$
$$\tilde{f}_s\big[x_{1000}(z_1)x_{0100}(m)x_{1100}(z_2)w_0ug\big]\psi(-r_1 + z_1)dr_idmdz_j.$$

By means of Fourier expansion this equals

$$\sum_{\eta\in F} \int_{(F\backslash\mathbb{A})^7} \varphi \left[\begin{pmatrix} 1 & r_1 & & \\ & 1 & & \\ & & 1 & -r_1 \\ & & & 1 \end{pmatrix} \begin{pmatrix} 1 & & r_2 & z_3 \\ & 1 & m & r_2 \\ & & 1 & \\ & & & 1 \end{pmatrix} g \right]$$
$$f_s\big[x_{1000}(z_1)x_{0100}(m+t)x_{1100}(z_2)w_0ug\big]\psi(-r_1 + z_1 + \eta t)dr_idmdz_jdt\ .$$

Provided $\eta \neq 0$, the integration over z_1, t and z_2 defines a Whittaker functional on the space of $\tilde{\theta}$. However it follows from [K-P] that such a functional is zero. On the other hand if $\eta = 0$, after a change of variables in t, we obtain as an inner integral

$$\int_{(F\backslash\mathbb{A})^3} \varphi \left[\begin{pmatrix} 1 & & r_2 & z_3 \\ & 1 & m & r_2 \\ & & 1 & \\ & & & 1 \end{pmatrix} h \right] dmdr_2dz_3$$

which is zero by cuspidality of φ. Thus the contribution of $\xi = 0$ to I is zero. Hence we may replace the sum over $\xi \in F$ by $\xi \in F^*$ and from the action of GL_1^Δ on $(0,\xi)$ we obtain

$$I = \int_{N(\mathbb{A})\backslash \mathrm{Sp}_4(\mathbb{A})} \int_{(F\backslash\mathbb{A})^6} \int_{U_0(\mathbb{A})} \varphi \left[\begin{pmatrix} 1 & r_1 & & \\ & 1 & & \\ & & 1 & -r_1 \\ & & & 1 \end{pmatrix} \begin{pmatrix} 1 & & r_2 & z_3 \\ & 1 & m & r_2 \\ & & 1 & \\ & & & 1 \end{pmatrix} g \right]$$
$$\omega_\psi \left[\begin{pmatrix} 1 & & & \\ & 1 & m & \\ & & 1 & \\ & & & 1 \end{pmatrix} \tau(u)g \right] \phi(0,1)$$
$$\tilde{f}_s\big[x_{1000}(z_1)x_{0100}(m)x_{1100}(z_2)w_0ug\big]\psi(-r_1 + z_1)dmdr_idz_jdudg\ .$$

Since

$$\omega_\psi\left[\begin{pmatrix}1&&&\\&1&m&\\&&1&\\&&&1\end{pmatrix}\tau(u)g\right]\phi(0,1)=\psi(1/2m)\omega_\psi\big(\tau(u)g\big)\phi(0,1)$$

this equals

$$I=\int_{N(\mathbf{A})\backslash \mathrm{Sp}_4(\mathbf{A})}\int_{(F\backslash\mathbf{A})^6}\int_{U_0(\mathbf{A})}\varphi\left[\begin{pmatrix}1&r_1&&\\&1&&\\&&1&-r_1\\&&&1\end{pmatrix}\begin{pmatrix}1&&r_2&z_3\\&1&m&r_2\\&&1&\\&&&1\end{pmatrix}g\right]$$
$$\omega_\psi\big(\tau(u)g\big)\phi(0,1)\tilde{f}_s\big[x_{1000}(z_1)x_{0100}(m)x_{1100}(z_2)w_0ug\big]$$
$$\psi(-r_1+z_1+1/2m)dmdr_idz_jdudg.$$

Write

$$\int_{(F\backslash\mathbf{A})^2}\tilde{f}_s\big[x_{1000}(z_1)x_{0100}(m)x_{1100}(z_2)w_0ug\big]\psi(z_1)dz_1dz_2$$
$$=\sum_{\eta\in F}\int_{(F\backslash\mathbf{A})^3}\tilde{f}_s\big[x_{1000}(z_1)x_{0100}(m+t)x_{1100}(z_2)w_0ug\big]\psi(z_1+\eta t)dz_idt\ .$$

As in the above, if $\eta\neq 0$ the integral vanishes. Thus

$$I=\int_{N(\mathbf{A})\backslash \mathrm{Sp}_4(\mathbf{A})}\int_{(F\backslash\mathbf{A})^7}\int_{U_0(\mathbf{A})}\varphi\left[\begin{pmatrix}1&r_1&&\\&1&&\\&&1&-r_1\\&&&1\end{pmatrix}\begin{pmatrix}1&&r_2&z_3\\&1&m&r_2\\&&1&\\&&&1\end{pmatrix}g\right]$$
$$\omega_\psi\big(\tau(u)g\big)\phi(0,1)\tilde{f}_s\big[x_{1000}(z_1)x_{0100}(m+t)x_{1100}(z_2)w_0\tau(u)g\big]$$
$$\psi(-r_1+1/2m+z_1)dmdtdr_idz_jdudg.$$

A change of variables in t and we are done. ∎

It follows from Theorem 2.1 that $I(\varphi,\phi,\tilde{f}_s,s)$ is factorizable. More precisely, write $\pi=\otimes\pi_\nu$, $\omega_\psi=\otimes\omega_\psi^{(\nu)}$, $\phi=\otimes\phi_\nu$ and $I(s)=\otimes I_\nu(s)$. If φ and $\tilde{f}_s$ are chosen so that $W_\varphi=\otimes W_\nu$ and $\tilde{f}_W=\otimes\tilde{f}_W^{(\nu)}$ then

$$I(\varphi,\phi,\tilde{f}_s,s)=\prod_\nu I_\nu(W_\nu,\phi_\nu,\tilde{f}_W^{(\nu)},s)$$

where

$$I_\nu(W_\nu,\phi_\nu,\tilde{f}_W^{(\nu)},s)=\int_{N(F_\nu)\backslash \mathrm{Sp}_4(F_\nu)}\int_{U_0(F_\nu)}W_\nu(g)\omega_\psi^{(\nu)}\big(\tau(u)g\big)$$
$$\phi_\nu(0,1)\tilde{f}_W^{(\nu)}(w_0ug,s)dudg.$$

This relation holds for $\mathrm{Re}(s)$ large.

3. The unramified computation

In this section we assume that F is a local nonarchimedean field. Let π be an irreducible admissible generic representation of Sp_4. Let ψ be a nontrivial additive character of F and denote by ω_ψ the Weil representation of $H_5 \cdot \widetilde{\mathrm{Sp}}_4$. This representation acts on $\mathcal{S}(F^2)$ as described in Section 1.3. Let $\tilde{\theta}$ denote the theta representation on $\widetilde{\mathrm{GL}}_3$ and denote by $\mathcal{W}(\tilde{\theta},\psi)$ its Whittaker model. Thus $\tilde{W}_\theta \in \mathcal{W}(\tilde{\theta},\psi)$ is a smooth function on $\widetilde{\mathrm{GL}}_3$ satisfying

$$\tilde{W}_\theta\left[\begin{pmatrix} 1 & x & y \\ & 1 & z \\ & & 1 \end{pmatrix} g\right] = \psi(x)\tilde{W}_\theta(g) \tag{3.1}$$

for $g \in \widetilde{\mathrm{GL}}_3$. Let $I(s) = \mathrm{Ind}_{P_0}^{\tilde{G}}\left(\delta_P^s \cdot (\gamma \circ \det) \otimes \mathcal{W}(\tilde{\theta},\psi)\right)$. Thus $\tilde{f}_W \in I(s)$ satisfies

$$\tilde{f}_W(gmv, s) = \tilde{W}_\theta(g)\delta_P^s(g)\gamma(\det g) \tag{3.2}$$

for $g \in \widetilde{\mathrm{GL}}_3$, $m \in \mathrm{Spin}_3$ and $v \in V$. Here γ is the local Weil constant. It is easy to check that $\delta_P^s\big(h(a_1,a_2,a_3,a_4)\big) = |a_3|^{5s}$.

Our local integral is

$$I(W,\phi,\tilde{f}_W,s) = \int_{N\backslash \mathrm{Sp}_4} \int_{U_0} W(g)\omega_\psi(\tau(u)g)\phi(0,1)\tilde{f}_W(w_0ug,s)\,du\,dg$$

where $\phi \in \mathcal{S}(F^2)$ and $w_0 = w(342341234)$. As in the global case we denote g for $j(g)$.

Let $(,)$ denote the local quadratic Hilbert symbol. We shall write γ_t for $\gamma(t)$ if $t \in F^*$. Thus $\gamma_a\gamma_b = (a,b)\gamma_{ab}$ and, since $(\varepsilon,\mu) = 1$, if ε,μ are units we have $\gamma_\varepsilon = 1$. The local computation of the integral involves some local calculations of $\sigma(g_1,g_2)$ in $\tilde{G}$. However, for most purposes we will need to know $\sigma(g_1,g_2)$ with $g_i \in M$. Identifying M with L (see Section 1.1) we may use the algorithm of computing the cocycle in GL_4, as described in [K-P] or more explicitly in [B-H], to compute $\sigma(g_1,g_2)$.

We shall denote by $\mathcal{O}$ the ring of integers in F and by $\mathcal{O}^*$ the units in $\mathcal{O}$. If p denotes a generator of the maximal ideal in $\mathcal{O}$ we let $q^{-1} = |p|$. All additive measures are chosen so that $\int_{\mathcal{O}} dx = 1$ and all multiplicative measures satisfy $\int_{\mathcal{O}^*} d^*x = 1$.

Let H be a reductive group. We denote by $K(H)$ its standard maximal compact subgroup. The group $K(G)$ splits in $\tilde{G}$. We shall describe the splitting homomorphism of $K(G)$ when restricted to an embedding of SL_2 corresponding to a simple root. For $k \in K(\mathrm{Spin}_3)$ we have $k \to \langle k, 1\rangle$. For $k \in K(\mathrm{GL}_2)$, where the GL_2 corresponds to one of the α_i, $1 \le i \le 3$, we have $k \to \langle k, \wedge(k)\rangle$ where

$$\wedge \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{cases} (c, d(ad-bc)) & 0 < |c| < 1 \\ 1 & |c| = 0, 1 \end{cases} \tag{3.3}$$

and $k = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in K(\mathrm{GL}_2)$ (see [K-P]).

In this section all functions are assumed to be unramified. Thus we assume that there exists $W \in \mathcal{W}(\pi, \psi)$ such that $W(k) = W(e) = 1$ for all $k \in K(\mathrm{Sp}_4)$. We also let ϕ denote the unramified vector in $\mathcal{S}(F^2)$. Thus $\phi(x_1, x_2) = 1$ if $|x_i| \le 1$ and zero otherwise. We let $\tilde{f}_W$ denote the $K(G)$ fixed vector in $I(s)$. This implies that $\tilde{W}_\theta$ is the $K(\mathrm{GL}_3)$ fixed vector in $\mathcal{W}(\tilde{\theta}, \psi)$ with $\tilde{W}_\theta(e) = 1$. Hence ψ is one on $\mathcal{O}$. We need to know the value of $\tilde{W}_\theta$ on the torus. From (3.1) it follows that $\tilde{W}_\theta(t) = 0$ unless

$$t = \begin{pmatrix} p^{m+n} & & \\ & p^n & \\ & & p^r \end{pmatrix} \begin{pmatrix} \varepsilon_1 & & \\ & \varepsilon_2 & \\ & & \varepsilon_3 \end{pmatrix}$$

with $m \ge 0$, $n, r \in \mathbb{Z}$ and $\varepsilon_i \in \mathcal{O}^*$. On t as above it follows from [B-G] that

$$\tilde{W}_\theta(t) = \begin{cases} \delta_{B_3}^{1/4}(t)\gamma_{p^n} & m \equiv 0(2) \\ 0 & m \equiv 1(2) \end{cases} \tag{3.4}$$

where B_3 denotes the standard Borel subgroup of GL_3 and δ_{B_3} its modular function. From (3.2) and (3.4) we obtain

(3.5)

$$\tilde{f}_W\big(h(a, a^2, a^2b, abc), s\big) = \delta_P^s\big(h(a, a^2, a^2b, abc)\big)\gamma_b \tilde{W}_\theta \begin{pmatrix} a & & \\ & a & \\ & & b \end{pmatrix}$$

$$= |a^2 b|^{5s} \gamma_b \tilde{W}_\theta \begin{pmatrix} a & & \\ & a & \\ & & b \end{pmatrix} = |a|^{10s+1/2} |b|^{5s-1/2} \gamma_a \gamma_b .$$

Next we describe the local adjoint L-function. By our assumption on π we may assume that $\pi = \mathrm{Ind}_B^{\mathrm{Sp}_4}(\mu_1, \mu_2)$ where B is the standard Borel subgroup of Sp_4 , i.e. $B \supset N$. We have

$$(\mu_1, \mu_2) \begin{pmatrix} a & & & * \\ & b & * & \\ & & b^{-1} & \\ & & & a^{-1} \end{pmatrix} = \mu_1(a)\mu_2(b)|a^4 b^2|.$$

From general theory we may associate to π a semisimple conjugacy class in $SO_5(\mathbb{C})$. We choose as a representative

$$t_\pi = \text{diag}\left(\mu_1(p), \mu_2(p), 1, \mu_2^{-1}(p), \mu_1^{-1}(p)\right).$$

Let Ad denote the adjoint representation of $SO_5(\mathbb{C})$. Its dimension is 10. We have

$$\begin{aligned} A(p) &= \text{Ad}(t_\pi) \\ &= \text{diag}\left(\mu_1(p)\mu_2(p), \mu_1(p)\mu_2^{-1}(p), \mu_1(p), \mu_2(p),\right. \\ &\quad \left. 1, 1, \mu_2^{-1}(p), \mu_1^{-1}(p), \mu_1^{-1}(p)\mu_2(p), \mu_1^{-1}(p)\mu_2^{-1}(p)\right). \end{aligned}$$

We define the local Adjoint L-function

$$L(\pi, \text{Ad}, s) = \det\left[I_{10} - A(p)q^{-s}\right]^{-1}$$

where I_{10} is the 10×10 identity matrix and $s \in \mathbb{C}$.

Finally we let $\zeta(s) = (1 - q^{-s})^{-1}$ denote the local zeta function. Our main Theorem:

THEOREM 3.1: *Assume that q is odd. For all unramified data and for* $\text{Re}(s)$ *large*

$$(3.6) \qquad I(W, \phi, \tilde{f}_W, s) = \frac{L(\pi, \text{Ad}, 5s-2)}{\zeta(10s-2)\zeta(10s-3)\zeta(10s-4)\zeta(20s-8)}.$$

Proof: Write I for $I(W, \phi, \tilde{f}_W, s)$. We start with the Iwasawa decomposition of Sp_4. Denote $h(a,b) = \text{diag}(a, b, b^{-1}, a^{-1})$. Via the embedding of Sp_4 in G, as explained in Section 1.1 we identify $h(a,b)$ in L with $h(a, ab, a, 1)$ in G.

Choosing the measure on $K(Sp_4)$ to be one, we obtain

$$\begin{aligned} I = \int_{(F^*)^2} \int_{U_0} & W\big(h(a,b)\big)\omega_\psi\big(\tau(u)h(a,b)\big)\phi(0,1) \\ & \tilde{f}_W\big(w_0 u h(a,ab,a,1), s\big)|a^4b^2|^{-1} du d^*a d^*b. \end{aligned}$$

The roots in U_0 are: (0001); (0011); (0111); (0012); (0112); (1112); (0122); (1122); (1222). We conjugate $h(a,b)$ across $\tau(u)$ and $h(a,ab,a,1)$ across u. A change of variables in U_0 will contribute $|a|^{-1}$ and hence

$$\begin{aligned} I = \int_{(F^*)^2} \int_{U_0} & W\big(h(a,b)\big)\omega_\psi\big(h(a,b)\tau(u)\big)\phi(0,1) \\ & \tilde{f}_W\big(w_0 h(a,ab,b,1)u, s\big)|a^5b^2|^{-1} du d^*a d^*b. \end{aligned}$$

From Section 1.1, i.e. from the description of the action of the Weyl group of G on its roots, we may deduce that $w_0 h(a,ab,b,1)w_0^{-1} = h(a,ab,a,a)$. We need to check if there is a cocycle contribution from this conjugation. Recall that $w_0 = w_3w_4w_2w_3w_4w_1w_2w_3w_4$, where w_i is the image of $\begin{pmatrix} & 1 \\ -1 & \end{pmatrix}$ in G. Since w_4 is in Spin_3, conjugation by w_4 will contribute no cocycle. We have (see Section 1.1) $w_4 \operatorname{diag}(a,b,b^{-1},a^{-1}) = \operatorname{diag}(a,b,b^{-1},a)$. Next consider the action of w_3. We have $w_3 \cdot \operatorname{diag}(a,b,b^{-1},a) = \operatorname{diag}(a,b,a,b^{-1})$. This conjugation will contribute the cocycle

$$\sigma\left[\begin{pmatrix} 1 &&& \\ & 1 && \\ && 0 & 1 \\ && -1 & 0 \end{pmatrix}, \begin{pmatrix} a &&& \\ & b && \\ && b^{-1} & \\ &&& a \end{pmatrix}\right]$$
$$\cdot\sigma\left[\begin{pmatrix} a &&& \\ & b && \\ && a & \\ &&& b^{-1} \end{pmatrix}, \begin{pmatrix} 1 &&& \\ & 1 && \\ && 0 & 1 \\ && -1 & 0 \end{pmatrix}\right].$$

We remind the reader (see Section 1.1) that by abuse of notation we denote by σ the cocycle on $\tilde{L}$ obtained by restriction from $\tilde{G}$. Using [B-H] the above cocycle equals $(a,b)(-1,a)$. The conjugation $w_2 \operatorname{diag}(a,b,a,b^{-1}) = \operatorname{diag}(a,a,b,b^{-1})$ contributes $(a,b)(-1,a)$. Next w_1, w_4 and w_3 each contributes one. Then w_2 contributes $(a,b)(-1,b)$ and finally w_3 also contributes $(a,b)(-1,b)$. Thus, overall, we get no contribution from the w_0 conjugation. Using Section 1.3 we obtain

$$I = \int_{(F^*)^2} \int_{U_0} W\big(h(a,b)\big)\omega_\psi\big(\tau(u)\big)\phi(0,b)$$
$$\tilde{f}_W\big(h(a,ab,a,a)w_0u,s\big)\gamma_{ab}(a,b)|a|^{-9/2}|b|^{-3/2}du d^*a d^*b.$$

Write $W\big(h(a,b)\big) = \delta_B^{1/2}\big(h(a,b)\big)K\big(h(a,b)\big)$. Then changing variables $a \to ab$ we get

$$I = \int_{(F^*)^2} \int_{U_0} K\big(h(ab,b)\big)\omega_\psi\big(\tau(u)\big)\phi(0,b)$$
$$\tilde{f}_W\big(h(ab,ab^2,ab,ab)w_0u,s\big)\gamma_a(ab,b)|a|^{-5/2}|b|^{-3}du d^*a d^*b.$$

The embedding of $h(ab,ab^2,ab,ab)$ in L is $\operatorname{diag}(ab,b,b^{-1},ab)$ and hence the equality $h(b,b^2,b,b)h(a,a,a,a) = h(ab,ab^2,ab,ab)$ contributes a symbol (a,b). Also, it follows from (3.5) that

$$\tilde{f}_W\big(h(b,b^2,b,b)g,s\big) = |b|^{5s+1}(b,b)\tilde{f}_W(g,s)$$

for all $g \in \tilde{G}$. Thus

$$I = \int_{(F^*)^2} \int_{U_0} K\big(h(ab,b)\big)\omega_\psi\big(\tau(u)\big)\phi(0,b)$$
$$\tilde{f}_W\big(H(a)w_0u,s\big)\gamma_a|a|^{-5/2}|b|^{5s-2}dud^*ad^*b$$

where $H(a) = h(a,a,a,a)$. From Section 1.3 we get, for $u \in U_0$,

$$\omega_\psi\big(\tau(u)\big)\phi(0,b) = \phi(x_1, b + x_2)\psi(z_1 + z_4 + by_1)$$

where $\tau(u) = (x_1, x_2, y_1, 0, z_1 + z_4)$ and

$$u = x_{0001}(x_1)x_{0011}(x_2)x_{0111}(y_1)x_{1112}(z_1)x_{0122}(z_4)u' .$$

From the properties of ϕ we obtain that if $|x_1| > 1$ the integral vanishes. Also $K\big(h(ab,b)\big) = 0$ if $|b| > 1$ and hence the condition $|b + x_2| \leq 1$ implies that $|x_2| \leq 1$. Thus the integration over x_1 and x_2 is restricted to $|x_1|, |x_2| \leq 1$. Let $U_1 \subset U_0$ be generated by the roots (0111); (0012); (0112); (1112); (0122); (1122); (1222). Using the right $K(G)$ invariance property we get

$$I = \int_{(F^*)^2} \int_{U_1} K\big(h(ab,b)\big)\psi(z_1 + z_4 + by_1)$$
$$\tilde{f}_W\big(H(a)w_0u,s\big)\gamma_a|a|^{-5/2}|b|^{5s-2}d^*ad^*bdu$$

where

(3.6) $$u = x_{0111}(y_1)x_{1112}(z_1)x_{1122}(z_2)x_{1222}(z_3)x_{0122}(z_4)x_{0112}(z_5)x_{0012}(z_6).$$

We claim that we may ignore the z_5, z_6 integration. To do that consider the function

$$F(z_5, z_6) = \int_{F^4} \tilde{f}_W\big(H(a)w_0u,s\big)\psi(z_1 + z_4)dz_1dz_2dz_3dz_4$$

where u is parameterized as in (3.6). Let $|t_1|, |t_2| \leq 1$. Then

$$F(z_5, z_6) = \int_{F^4} \tilde{f}_W\big(H(a)w_0ux_{1000}(t_1)x_{1100}(t_2),s\big)\psi(z_1 + z_4)dz_1dz_2dz_3dz_4$$
$$= \int_{F^4} \tilde{f}_W\big(H(a)w_0x_{1000}(t_1)x_{1100}(t_2)u,s\big)\psi(z_1 + z_4 - t_1z_5 - t_2z_6)dz_1dz_2dz_3dz_4$$
$$= \psi(-t_1z_5 - t_2z_6)F(z_5, z_6).$$

Here we used a change of variables in u and the relations $w_0x_{1000}w_0^{-1} = x_{0012}$ and $w_0x_{1100}w_0^{-1} = x_{0112}$. Since $\tilde{f}_W$ is left invariant by x_{0012} and x_{0112} the last equality follows. Hence $F(z_5, z_6) = 0$ if $|z_5| > 1$ or $|z_6| > 1$. Write $w_0 = \overline{w}w(4234)$ where $\overline{w} = w(32143)$. Hence

$$I = \int_{(F^*)^2} \int_{F^5} K\big(h(ab, b)\big)$$
$$\tilde{f}_W\big(H(a)\overline{w}x_{1000}(z_1)x_{1110}(z_2)x_{1122}(z_3)x_{0010}(z_4)x_{0011}(-y_1), s\big)$$
$$\psi(z_1 + z_4 + by_1)\gamma_a|a|^{-5/2}|b|^{5s-2}dz_idy_1d^*ad^*b.$$

We know that $K\big(h(ab,b)\big) = 0$ if $|a| > 1$ or $|b| > 1$. For $|a| \leq 1$ and $|b| \leq 1$ write $a = p^n\varepsilon_1$ and $b = p^m\varepsilon_2$ with $n, m \geq 0$. Since $H(p^n\varepsilon) = H(p^n)H(\varepsilon)$ contributes a symbol (p^n, ε) which cancels with (p^n, ε) obtained from the relation $\gamma_{p^n\varepsilon} = \gamma_{p^n}(p^n, \varepsilon)$, we get

$$I = \sum_{n,m=0}^{\infty} K\big(h(p^{n+m}, p^m)\big)J(n,m)\gamma_{p^n}q^{5n/2+(-5s+2)m}$$

where

$$J(n,m) = \int_{F^5} \tilde{f}_W\big(H(p^n)\overline{w}x_{1000}(z_1)x_{1110}(z_2)x_{1122}(z_3)x_{0010}(z_4)x_{0011}(-y_1), s\big)$$
$$\psi(z_1 + z_4 + p^my_1)dz_idy_1 .$$

We need to compute $J(n,m)$. Define, for $n \geq 0$,

$$J(n) = \int_{F^3} \tilde{f}_W\big(H(p^n)w(321)x_{1000}(z_1)x_{1100}(z_2)x_{1110}(z_3), s\big)\psi(z_1)dz_i$$

and let $J(n) = 0$ for $n < 0$. Also, let

$$G(p,\psi) = \sum_{\varepsilon\in(\mathcal{O}/\mathcal{P})^*} (p,\varepsilon)\psi(p^{-1}\varepsilon)$$

where $\mathcal{P}$ denotes the maximal ideal in $\mathcal{O}$. We have

LEMMA 3.2: *For* $n, m \geq 0$

$$J(n,m) = \frac{\zeta(10s-4)}{\zeta(10s-3)}\big[(1 - q^{(-10s+4)(m+1)})J(n)$$
$$+ q^{-15s+3}(p, p^{n-1})G(p,\psi)(1 - q^{(-10s+4)m})J(n-1)\big].$$

Proof: To prove the Lemma we use the Iwasawa decomposition for z_4 first and then y_1. Before going into details, recall the following SL_2 decomposition:

$$\begin{pmatrix} & 1 \\ -1 & \end{pmatrix}\begin{pmatrix} 1 & r \\ & 1 \end{pmatrix} = \begin{pmatrix} r^{-1} & -1 \\ & r \end{pmatrix}\begin{pmatrix} -1 & \\ -r^{-1} & -1 \end{pmatrix}. \tag{3.7}$$

The symbol coming out from this multiplication is one (see [B-H]), however since $\wedge\begin{pmatrix} -1 & \\ -r^{-1} & -1 \end{pmatrix} = (r,r)$, there is a symbol contribution of (r,r) from this factorization. Thus

$$\begin{aligned} J(n,m) = & \int_{F^5} \tilde{f}_W \big[H(p^n)w(3214)x_{1000}(z_1)x_{1100}(z_2)x_{1112}(z_3) \\ & \qquad x_{0001}(y_1)w(3)x_{0010}(z_4), s\big] \\ & \cdot \psi(z_1+z_4+p^m y_1)dz_i dy_1 \\ = & \int_{F^4} \tilde{f}_W \big[H(p^n)w(3214)x_{1000}(z_1)x_{1100}(z_2)x_{1112}(z_3)x_{0001}(y_1), s\big] \\ & \cdot \psi(z_1+p^m y_1)dz_i dy_1 \\ & + \int_{F^4}\int_{|z_4|>1} \tilde{f}_W \big[H(p^n)w(3214)x_{1000}(z_1)x_{1100}(z_2)x_{1112}(z_3) \\ & x_{0001}(y_1)x_{0010}(-z_4^{-1})h(1,1,z_4^{-1},1), s\big](z_4,z_4)\psi(z_1+z_4+p^m y_1)dz_i dy_1 \end{aligned}$$

where the last equality comes from breaking the z_4 integration into $|z_4| \le 1$ and $|z_4| > 1$, and (z_4, z_4) is contributed from the Iwasawa decomposition of $w(3)x_{0010}(z_4)$ as explained in (3.7). Write I_1 (resp. I_2) for the first (resp. second) summand. In I_1 we separate the y_1 integration into $|y_1| \le 1$ and $|y_1| > 1$. In the $|y_1| > 1$ domain we also perform the Iwasawa decomposition (3.7) for the root $x_{0001}(y_1)$. Since this root is in Spin_3, no symbol is added. Thus

$$\begin{aligned} I_1 = & \int_{F^3} \tilde{f}_W \big(H(p^n)w(321)x_{1000}(z_1)x_{1100}(z_2)x_{1110}(z_3), s\big)\psi(z_1)dz_i \\ & + \int_{F^3}\int_{|y_1|>1} \tilde{f}_W \big(H(p^n)w(321)x_{1000}(z_1) \\ & x_{1100}(z_2)x_{1110}(z_3)x_{0001}(-y_1^{-1})h(1,1,1,y_1^{-1}), s\big)\psi(z_1+p^m y_1)dz_i dy_1 . \end{aligned}$$

We used the fact that since $m \ge 0$, $p^m y_1 \in \mathcal{O}$ if $|y_1| \le 1$. Notice that the first term is $J(n)$. In the second term we may conjugate $x_{0001}(-y_1^{-1})$ across and then conjugate $h(1,1,1,y_1^{-1})$. We get

$$\begin{aligned} I_1 = J(n) + & \int_{F^3}\int_{|y_1|>1} \tilde{f}_W \big(H(p^n)h(1,1,y_1^{-2},y_1)w(321) \\ & x_{1000}(z_1)x_{1100}(z_2)x_{1110}(z_3), s\big)\psi(z_1+p^m y_1)|y_1|^2 dz_i dy_1 . \end{aligned}$$

We used the relation (which contributes no cocycle)

$$w(321)h(1,1,1,y_1^{-1})w(321)^{-1} = h(1,1,y_1^{-2},y_1)$$

and the $|y_1|^2$ appears from the change of variables $z_3 \to y_1^2 z_3$ which arises from the conjugation of $h(1,1,1,y_1^{-1})$ across the unipotent matrices. We have from (3.5)

$$\tilde{f}_W\big(h(1,1,y_1^{-2},y_1)g,s\big) = |y_1|^{-10s+1}\tilde{f}_W(g,s)\ .$$

Thus

$$\begin{aligned} I_1 &= J(n) + \int_{|y_1|>1} |y_1|^{-10s+3}\psi(p^m y_1)dy_1 \\ &\quad \cdot \int_{F^3} \tilde{f}_W\big(H(p^n)w(321)x_{1000}(z_1)x_{1100}(z_2)x_{1110}(z_3),s\big)\psi(z_1)dz_i \\ &= L(m)J(n) \end{aligned}$$

where

$$L(m) = 1 + \int_{|y_1|>1} |y_1|^{-10s+3}\psi(p^m y_1)dy_1\ .$$

It follows from [G1] that

$$L(m) = \frac{\zeta(10s-4)}{\zeta(10s-3)}\left(1 - q^{(-10s+4)(m+1)}\right)\ .$$

Next we consider I_2. We conjugate $x_{0010}(-z_4^{-1})$ and $h(1,1,z_4^{-1},1)$. The torus $h(1,1,z_4^{-1},1)$ is identified with $\mathrm{diag}(1,1,z_4^{-1},z_4)$ in L and one can check that no cocycle is contributed from the relation

$$w(3214)h(1,1,z_4^{-1},1)w(3214)^{-1} = h(1,z_4^{-1},z_4^{-2},z_4^{-1})$$

and hence

$$\begin{aligned} I_2 = \int_{F^4}\int_{|z_4|>1} &\tilde{f}_W\big(H(p^n)h(1,z_4^{-1},z_4^{-2},z_4^{-1})w(3214) \\ &x_{1000}(z_1)x_{1100}(z_2)x_{1112}(z_3)x_{0001}(y_1),s\big) \\ &(z_4,z_4)|z_4|^3\psi(z_1+z_4+p^m y_1)dz_i dy_1 \end{aligned}$$

where $|z_4|^3$ is obtained from change of variables.

Write $H(p^n)h(1, z_4^{-1}, z_4^{-2}, z_4^{-1}) = h(z_4^{-1}, z_4^{-2}, z_4^{-3}, z_4^{-4})H(z_4p^n)$. This relation contributes the cocycle (p^nz_4, z_4). Thus using (3.5)

$$I_2 = \int_{F^4} \int_{|z_4|>1} \tilde{f}_W\left(H(z_4p^n)w(3214)x_{1000}(z_1)x_{1100}(z_2)x_{1112}(z_3)x_{0001}(y_1), s\right)$$
$$\psi(z_1 + z_4 + p^m z_4 y_1)(z_4p^n, z_4)|z_4|^{-15s+3} dz_i dy_1 \ .$$

Next we break the y_1 integration into $|y_1| \le 1$ and $|y_1| > 1$. We obtain

$$I_2 = \int_{F^3} \int_{|z_4|>1} \tilde{f}_W\left(H(z_4p^n)w(321)x_{1000}(z_1)x_{1100}(z_2)x_{1110}(z_3), s\right)$$
$$\psi(z_1 + z_4)|z_4|^{-15s+3}(z_4, p^nz_4)\left(\int_{|y_1|\le 1} \psi(p^m z_4 y_1)dy_1\right) dz_i$$
$$+ \int_{F^3} \int_{|z_4|,|y_1|>1} \tilde{f}_W\left(H(z_4p^n)w(321)x_{1000}(z_1)\right.$$
$$\left.x_{1100}(z_2)x_{1110}(z_3)w_4x_{0001}(y_1), s\right)$$
$$\psi(z_1 + z_4 + p^m z_4 y_1)|z_4|^{-15s+3}(z_4, p^nz_4)dy_1dz_i \ .$$

In the second term we write the Iwasawa decomposition for $w_4x_{0001}(y_1)$. Then repeating the same steps as in the computation of I_1 we obtain

$$I_2 = \int_{F^3} \int_{|z_4|>1} \tilde{f}_W\left(H(z_4p^m)w(321)x_{1000}(z_1)x_{1100}(z_2)x_{1110}(z_3), s\right)$$
$$\psi(z_1 + z_4)|z_4|^{-15s+3}(z_4, p^nz_4)$$
$$\left(\int_{|y_1|\le 1} \psi(p^m z_4 y_1)dy_1 + \int_{|y_1|>1} |y_1|^{-10s+3}\psi(p^m z_4 y_1)dy_1\right) dz_i \ .$$

Write $\int_{|z_4|>1} = \sum_{r=1}^{\infty} \int_{|\varepsilon|=1}$. Thus

$$I_2 = \int_{F^3} \sum_{r=1}^{\infty} q^r \int_{|\varepsilon|=1} \tilde{f}_W\left(H(p^{n-r}\varepsilon)w(321)x_{1000}(z_1)x_{1100}(z_2)x_{1110}(z_3), s\right)$$
$$\psi(z_1 + p^{-r}\varepsilon)q^{(-15s+3)r}(p^{-r}\varepsilon, p^{n-r}\varepsilon)$$
$$\left(\int_{|y_1|\le 1} \psi(p^{m-r}y_1)dy_1 + \int_{|y_1|>1} |y_1|^{-10s+3}\psi(p^{m-r}y_1)dy_1\right) dz_i.$$

The factorization $H(p^{n-r}\varepsilon) = H(p^{n-r})H(\varepsilon)$ gives a (p^{n-r}, ε) contribution. Conjugating $H(\varepsilon)$ to the right and using the $K(G)$ right invariance property of $\tilde{f}_W$, we obtain

$$I_2 = \sum_{r=1}^{\infty} J(n-r) q^{(-15s+4)r} (p^r, p^{n-r})$$
$$\left(\int_{|y| \le 1} \psi(p^{m-r} y_1) dy_1 + \int_{|y_1|>1} |y_1|^{-10s+3} \psi(p^{m-r} y_1) dy_1 \right)$$
$$\int_{|\varepsilon|=1} (p^r, \varepsilon) \psi(p^{-r}\varepsilon) d\varepsilon \ .$$

The last integral is zero unless $r = 1$ and for $r = 1$ it equals $q^{-1}G(p,\psi)$. Thus

$$I_2 = J(n-1) q^{(-15s+3)r} (p, p^{n-1}) G(p,\psi)$$
$$\left(\int_{|y_1| \le 1} \psi(p^{m-1} y_1) dy_1 + \int_{|y_1|>1} |y_1|^{-10s+3} \psi(p^{m-1} y_1) dy_1 \right) .$$

It is easy to see that if $m = 0$ the sum of the integrals equals zero and for $m > 0$ we get $L(m-1)$. The Lemma follows. ∎

We compute $J(n)$. Write $J(n) = R_1(n) + R_2(n)$ where

$$R_1(n) = \int_{F^2} \tilde{f}_W \big(H(p^n) w(32) x_{0100}(z_2) x_{0110}(z_3), s \big) dz_i$$

and

$$R_2(n) = \int_{F^2} \int_{|z_1|>1} \tilde{f}_W \big(H(p^n) w(321) x_{1000}(z_1) x_{1100}(z_2) x_{1110}(z_3), s \big) \psi(z_1) dz_i \ .$$

For $\ell \ge 0$, let

$$M(\ell) = \int_{F^2} \tilde{f}_W \big(H(p^\ell) w(321) x_{1000}(p^{-1}) x_{1100}(z_2) x_{1110}(z_3), s \big) dz_i \ .$$

We have

LEMMA 3.3: *For $k \ge 0$*

$$J(2k) = R_1(2k) - M(2k),$$
$$J(2k+1) = G(p,\psi) M(2k+1) \ .$$

Proof: Let us show that $R_1(2k+1) = 0$. Indeed, change variables $z_2 \to \varepsilon z_2$ with $|\varepsilon| = 1$. Using the relation

$$h(1,\varepsilon,\varepsilon,\varepsilon) x_{0100}(z_2) h(1,\varepsilon,\varepsilon,\varepsilon)^{-1} = x_{0100}(\varepsilon z_2)$$

we obtain by the right invariance properties of $\tilde{f}_W$

$$\begin{aligned} R_1(n) &= \int_{F^2} \tilde{f}_W\big(H(p^n)w(32)h(1,\varepsilon,\varepsilon,\varepsilon)x_{0100}(z_2)x_{0110}(z_3),s\big)dz_i \\ &= \int_{F^2} \tilde{f}_W\big(H(p^n)h(1,1,\varepsilon,\varepsilon)w(32)x_{0100}(z_2)x_{0110}(z_3),s\big)dz_i \\ &= (p^n,\varepsilon)\int_{F^2} \tilde{f}_W\big(H(p^n)w(32)x_{0100}(z_2)x_{0110}(z_3),s\big)dz_i \\ &= (p^n,\varepsilon)R_1(n)\ . \end{aligned}$$

Here the cocycle (p^n,ε) is obtained from the conjugation of the tori elements. Also we used (3.5). Thus $R_1(2k+1)=0$.

Next consider $R_2(n)$. We have

$$\begin{aligned} R_2(n) = \int_{F^2}\sum_{r=1}^{\infty} q^r \int_{|\varepsilon|=1} & \tilde{f}_W\big(H(p^n)w(321)x_{1000}(p^{-r}\varepsilon) \\ & \cdot x_{1100}(z_2)x_{1110}(z_3),s\big)\psi(p^{-r}\varepsilon)d\varepsilon dz_i . \end{aligned}$$

We have

$$h(\varepsilon,\varepsilon,\varepsilon,1)x_{1000}(p^{-r})h(\varepsilon,\varepsilon,\varepsilon,1)^{-1} = x_{1000}(p^{-r}\varepsilon)$$

and

$$w(321)h(\varepsilon,\varepsilon,\varepsilon,1)w(321)^{-1} = h(1,1,\varepsilon,\varepsilon)\ .$$

As before the conjugation of $H(p^n)$ with $h(1,1,\varepsilon,\varepsilon)$ contributes a cocycle (p^n,ε). Thus

$$\begin{aligned} R_2(n) = \int_{F^2}\sum_{r=1}^{\infty} q^r \tilde{f}_W\big(H(p^n)w(321)x_{1000}(p^{-1})x_{1100}(z_2)x_{1110}(z_3),s\big)dz_i \\ \int_{|\varepsilon|=1}(p^n,\varepsilon)\psi(p^{-r}\varepsilon)d\varepsilon. \end{aligned}$$

The last integral is zero if $r>1$. For $r=1$ we have (see [G2])

$$\int_{|\varepsilon|=1}(p^n,\varepsilon)\psi(p^{-1}\varepsilon)d\varepsilon = \begin{cases} -q^{-1} & n\equiv 0(2), \\ q^{-1}G(p,\psi) & n\equiv 1(2). \end{cases}$$

From this the Lemma follows. ∎

We continue with the computation of $M(\ell)$.

LEMMA 3.4: *We have*

$$M(2k)=\frac{\zeta(10s-3)}{\zeta(10s-2)}q^{(-20s+3)k-10s+3}$$

and

$$M(2k+1)=\frac{\zeta(10s-4)}{\zeta(10s-2)}q^{(-10s-1)k-10s+1}\left(1-q^{(-10s+4)(k+1)}\right) .$$

Proof: Write the Iwasawa decomposition for $w_1x_{1000}(p^{-1})$ (see (3.7)). We get

$$M(\ell)=(p,p)\int_{F^2}\tilde{f}_W\left(H(p^\ell)w(32)x_{0100}(z_2)x_{0110}(z_3)x_{1000}(p)h(p,1,1,1),s\right)dz_i$$

where the (p,p) is obtained from this Iwasawa decomposition. Conjugating $x_{1000}(p)$ across the unipotent elements we get

$$x_{0100}(z_2)x_{0110}(z_3)x_{1000}(p)=x_{1100}(pz_2)ux_{0100}(z_2)x_{0110}(z_3)$$

where u is such that

$$\tilde{f}_W\left(H(p^\ell)w(32)ug,s\right)=\tilde{f}_W\left(H(p^\ell)w(32)g,s\right) .$$

Since $w(32)x_{1100}(pz_2)w(32)^{-1}=x_{1000}(pz_2)$ we obtain using (3.1)

$$M(\ell)=(p,p)\int_{F^2}\tilde{f}_W\left(H(p^\ell)w(32)x_{0100}(z_2)x_{0110}(z_3)h(p,1,1,1),s\right)\psi(p^{\ell+1}z_2)dz_i .$$

Conjugating the torus across we get

$$M(\ell)=(p^{\ell+1},p)q^2\int_{F^2}\tilde{f}_W\left(h(p^{\ell+1},p^{\ell+1},p^{\ell+1},p^\ell)w(32)x_{0100}(z_2)x_{0110}(z_3),s\right)$$
$$\psi(p^\ell z_2)dz_i.$$

The (p^ℓ,p) factor is from the product $H(p^\ell)h(p,p,p,1)=h(p^{\ell+1},p^{\ell+1},p^{\ell+1},p^\ell)$ and q^2 is obtained from changing variables in z_2 and z_3. Separating the z_2

integration into $|z_2| \le 1$ and $|z_2| > 1$, we obtain

$$\begin{aligned}
M(\ell) = (p^{\ell+1},p)q^2 \Bigg[&\int_F \tilde{f}_W\big(h(p^{\ell+1},p^{\ell+1},p^{\ell+1},p^{\ell})w(3)x_{0010}(z_3),s\big)dz_3 \\
&+ \int_F \int_{|z_2|>1} \tilde{f}_W\big(h(p^{\ell+1},p^{\ell+1},p^{\ell+1},p^{\ell})w(3)x_{0010}(z_3)w(2)x_{0100}(z_2),s\big) \\
&\cdot \psi(p^{\ell}z_2)dz_i\Bigg] \\
= (p^{\ell+1},p)q^2 \Bigg[&\tilde{f}_W\big(h(p^{\ell+1},p^{\ell+1},p^{\ell+1},p^{\ell}),s\big) \\
&+ \int_{|z_3|>1} \tilde{f}_W\big(h(p^{\ell+1},p^{\ell+1},p^{\ell+1},p^{\ell})h(1,1,z_3^{-1},1),s\big)(z_3,z_3)dz_3 \\
&+ \int_F \sum_{r=1}^{\infty} q^r \int_{|\varepsilon|=1} \tilde{f}_W\big(h(p^{\ell+1},p^{\ell+1},p^{\ell+1},p^{\ell})w(3)x_{0010}(z_3)w(2)x_{0100}(p^{-r}\varepsilon),s\big) \\
&\cdot \psi(p^{\ell-r}\varepsilon)d\varepsilon dz_3\Bigg].
\end{aligned}$$

Here we used the Iwasawa decomposition for z_3 and also $\int_{|z_2|>1} = \sum_{r=1}^{\infty}\int_{|\varepsilon|=1}$. Thus,

$$\begin{aligned}
M(\ell) = (p^{\ell+1},p)q^2 \Bigg[&\tilde{f}_W\big(h(p^{\ell+1},p^{\ell+1},p^{\ell+1},p^{\ell}),s\big) \\
&\left(1 + \int_{|z_3|>1} (z_3, z_3p^{\ell-1})\gamma_{z_3^{-1}}|z_3|^{-5s+1/2}dz_3\right) + \\
&\int_F \sum_{r=1}^{\infty} q^r \tilde{f}_W\big(h(p^{\ell+1},p^{\ell+1},p^{\ell+1},p^{\ell})w(3)x_{0010}(z_3)w(2)x_{0100}(p^{-r}),s\big)dz_3 \\
&\int_{|\varepsilon|=1}(p^{\ell-1},\varepsilon)\psi(p^{\ell-r}\varepsilon)d\varepsilon\Bigg].
\end{aligned}$$

Suppose $\ell = 2k$. From (3.4) it follows that $\tilde{f}_W\big(h(p^{2k+1},p^{2k+1},p^{2k+1},p^{2k}),s\big) = 0$. Thus

$$\begin{aligned}
M(2k) = (p,p)q^2 \int_F \sum_{r=1}^{\infty} q^r \tilde{f}_W\big(&h(p^{2k+1},p^{2k+1},p^{2k+1},p^{2k}) \\
&\cdot w(3)x_{0010}(z_3)w(2)x_{0100}(p^{-r}),s\big)dz_3 \int_{|\varepsilon|=1}(p,\varepsilon)\psi(p^{2k-r}\varepsilon)d\varepsilon.
\end{aligned}$$

The last integral vanishes unless $r = 2k+1$. Thus

$$M(2k) = (p,p)q^{2k+2}G(p,\psi)$$
$$\int_F \tilde{f}_W\big(h(p^{2k+1},p^{2k+1},p^{2k+1},p^{2k})w(3)x_{0010}(z_3)w(2)x_{0100}(p^{-2k-1}),s\big)dz_3$$
$$= q^{2k+2}G(p,\psi)$$
$$\int_F \tilde{f}_W\big(h(p^{2k+1},p^{2k+1},p^{2k+1},p^{2k})w(3)x_{0010}(z_3)h(1,p^{2k+1},1,1),s\big)dz_3.$$

Here the (p,p) factor is cancelled by (p^{2k+1},p^{2k+1}) obtained from the Iwasawa decomposition. Thus

$$M(2k) = q^{4k+3}G(p,\psi)\int_F \tilde{f}_W\big(h(p^{2k+1},p^{4k+2},p^{4k+2},p^{2k})w(3)x_{0010}(z_3),s\big)dz_3$$
$$= q^{4k+3}G(p,\psi)\gamma_{p^{2k+1}}q^{-(4k+2)5s-k-1/2}\int_F \tilde{f}_W\big(w(3)x_{0010}(z_3),s\big)dz_3$$

where we used (3.4) and (3.5). It follows from [R-S] that $\gamma_p G(p,\psi) = q^{1/2}$. Hence

$$M(2k) = q^{(-20s+3)k-10s+3}\left(1+\int_{|z_3|>1}\tilde{f}_W\big(h(1,1,z_3^{-1},1),s\big)(z_3,z_3)dz_3\right)$$
$$= q^{(-20s+3)k-10s+3}\left(1+\int_{|z_3|>1}(z_3,z_3)\gamma_{z_3^{-1}}|z_3|^{-5s+1/2}dz_3\right)$$
$$= q^{(-20s+3)k-10s+3}\left(1+\sum_{r=1}^{\infty}(p^{-r},p^{-r})q^{(-5s+3/2)r}\int_{|\varepsilon|=1}\gamma_{p^{-r}\varepsilon}d\varepsilon\right).$$

The last integral is zero unless r is even, and in this case it equals $1-q^{-1}$. Thus

$$M(2k) = q^{(-20s+3)k-10s+3}\left(1+(1-q^{-1})\sum_{r=1}^{\infty}q^{(-10s+3)r}\right)$$
$$= \frac{\zeta(10s-3)}{\zeta(10s-2)}q^{(-20s+3)k-10s+3}.$$

Next we consider the case $\ell = 2k+1$. We have

$$M(2k+1) = q^2\Bigg[\tilde{f}_W\big(h(p^{2k+2},p^{2k+2},p^{2k+2},p^{2k+1}),s\big)$$
$$\cdot\left(1+\int_{|z_3|>1}(z_3,z_3)\gamma_{z_3^{-1}}|z_3|^{-5s+1/2}dz_3\right)+$$
$$\int_F\sum_{r=1}^{\infty}q^r\tilde{f}_W\big(h(p^{2k+2},p^{2k+2},p^{2k+2},p^{2k+1})w(3)x_{0010}(z_3)w(2)x_{0100}(p^{-r}),s\big)dz_3$$
$$\cdot\int_{|\varepsilon|=1}\psi(p^{2k-r+1}\varepsilon)d\varepsilon\Bigg].$$

From (3.2) it follows that

$$\tilde{f}_W\big(h(p^{2k+2},p^{2k+2},p^{2k+2},p^{2k+1}),s\big) = q^{-(2k+2)5s-k-1}\ .$$

Also, as in the computation for $M(2k)$

$$1+\int_{|z_3|>1}(z_3,z_3)\gamma_{z_3^{-1}}|z_3|^{-5s+1/2}dz_3 = \frac{\zeta(10s-3)}{\zeta(10s-2)}\ .$$

The integral

$$\int_{|\varepsilon|=1}\psi(p^{2k-r+1}\varepsilon)d\varepsilon = \begin{cases} 1-q^{-1} & r\le 2k+1\\ -q^{-1} & r=2k+2\\ 0 & r>2k+2\end{cases}$$

Thus

$$\begin{aligned} M(2k+1) =&q^2\Bigg[\frac{\zeta(10s-3)}{\zeta(10s-2)}q^{-(2k+2)5s-k-1}\\ &+\int_F\sum_{r=1}^{2k+1}q^r(1-q^{-1})\tilde{f}_W\big(h(p^{2k+2},p^{2k+2},p^{2k+2},p^{2k+1})\\ &w(3)x_{0010}(z_3)w(2)x_{0100}(p^{-r}),s\big)dz_3\\ &-q^{2k+1}\int_F\tilde{f}_W\big(h(p^{2k+2},p^{2k+2},p^{2k+2},p^{2k+1})\\ &w(3)x_{0010}(z_3)w(2)x_{0100}(p^{-2k-2}),s\big)dz_3\Bigg].\end{aligned}$$

Next we write the Iwasawa decomposition for $w(2)x_{0100}(p^{-r})$ for $1\le r\le 2k+2$.

We get

$$M(2k+1) = q^2 \Bigg[\frac{\zeta(10s-3)}{\zeta(10s-2)} q^{-(2k+2)5s-k-1}$$

$$+ (1-q^{-1}) \sum_{r=1}^{2k+1} q^r (p^r, p^r) \int_F \tilde{f}_W \big(h(p^{2k+2}, p^{2k+2}, p^{2k+2}, p^{2k+1})$$

$$w(3) x_{0010}(z_3) h(1, p^r, 1, 1), s\big) dz_3 - q^{2k+1} \int_F \tilde{f}_W \big(h(p^{2k+2}, p^{2k+2}, p^{2k+2}, p^{2k+1})$$

$$w(3) x_{0010}(z_3) h(1, p^{2k+2}, 1, 1), s\big) dz_3 \Bigg]$$

$$= q^2 \Bigg[\frac{\zeta(10s-3)}{\zeta(10s-2)} q^{-(2k+2)5s-k-1}$$

$$+ (1-q^{-1}) \sum_{r=1}^{2k+1} q^{2r} (p^r, p^r) \int_F \tilde{f}_W \big(h(p^{2k+2}, p^{2k+r+2}, p^{2k+r+2}, p^{2k+1})$$

$$\cdot w(3) x_{0010}(z_3), s\big) dz_3$$

$$- q^{4k+3} \int_F \tilde{f}_W \big(h(p^{2k+2}, p^{4k+4}, p^{4k+4}, p^{2k+1}) w(3) x_{0010}(z_3), s\big) dz_3 \Bigg].$$

We have, using (3.5), that

$$\tilde{f}_W \big(h(p^{2k+2}, p^{4k+4}, p^{4k+4}, p^{2k+1}), s\big) = q^{-(4k+4)5s-k-1}.$$

Also, as before in the case of $M(2k)$, we have

$$\int_F \tilde{f}_W \big(w(3) x_{0010}(z_3), s\big) dz_3 = \frac{\zeta(10s-3)}{\zeta(10s-2)}.$$

Thus the third summand equals

$$-\frac{\zeta(10s-3)}{\zeta(10s-2)} q^{(-20s+3)k-20s+2}.$$

In the second summand we claim that r is even. Indeed, this follows from (3.4). Also since the GL_2 which corresponds to the root α_3 commutes with x_{1000}, the z_3 integration does not affect this statement about r. Thus, the second summand equals

$$(1-q^{-1})\sum_{r=1}^{k} q^{4r}\int_F \tilde{f}_W\big(h(p^{2k+2},p^{2k+2r+2},p^{2k+2r+2},p^{2k+1})w(3)x_{0010}(z_3),s\big)dz_3$$
$$=(1-q^{-1})\frac{\zeta(10s-3)}{\zeta(10s-2)}\sum_{r=1}^{k} q^{4r}\tilde{f}_W\big(h(p^{2k+2},p^{2k+2r+2},p^{2k+2r+2},p^{2k+1}),s\big)$$
$$=(1-q^{-1})\frac{\zeta(10s-3)}{\zeta(10s-2)}\sum_{r=1}^{k} q^{4r-(2k+2r+2)5s-k-1}\ .$$

Combining all pieces, we obtain

$$M(2k+1)=$$
$$q^2\frac{\zeta(10s-3)}{\zeta(10s-2)}\Bigg[q^{(-10s-1)(k+1)} + (1-q^{-1})q^{(-10s-1)k-10s-1}\sum_{r=1}^{k} q^{(-10s+4)r} - q^{(-20s+3)k-20s+2}\Bigg].$$

From this the Lemma follows. ∎

Finally, we have,

LEMMA 3.5: *For $k\geq 0$*

$$J(2k)=\frac{\zeta(10s-4)}{\zeta(10s-2)}q^{(-10s-1)k}(1-q^{(-10s+4)(k+1)})$$

and

$$J(2k+1)=\frac{\zeta(10s-4)}{\zeta(10s-2)}G(p,\psi)q^{(-10s-1)k-10s+1}(1-q^{(-10s+4)(k+1)})\ .$$

Proof: From Lemmas 3.3 and 3.4 it is enough to compute

$$R_1(2k)=\int_{F^2}\tilde{f}_W\big(H(p^{2k})w(32)x_{0100}(z_2)x_{0110}(z_3),s\big)dz_i\ .$$

Separating the z_2 integration to $|z_2|\leq 1$ and $|z_2|>1$, and performing the Iwasawa decomposition when $|z_2|>1$, we obtain

$$R_1(2k)=\int_F \tilde{f}_W\big(H(p^{2k})w(3)x_{0010}(z_3),s\big)dz_3$$
$$+\int_F\int_{|z_2|>1}\tilde{f}_W\big(H(p^{2k})w(3)x_{0010}(z_3)h(1,z_2^{-1},1,1),s\big)(z_2,z_2)dz_i\ .$$

In the first summand, as in the computation of $M(2k)$, we may perform the z_3 integration to get

$$\frac{\zeta(10s-3)}{\zeta(10s-2)}\tilde{f}_W\left(H(p^{2k})\right)=\frac{\zeta(10s-3)}{\zeta(10s-2)}q^{(-10s-1)k}\ .$$

Conjugating the torus $h(1,z_2^{-1},1,1)$ in the second summand, we obtain

$$\int_F\int_{|z_2|>1}\tilde{f}_W\left(h(p^{2k},p^{2k}z_2^{-1},p^{2k}z_2^{-1},p^{2k})w(3)x_{0010}(z_3),s\right)(z_2,z_2)|z_2|dz_i\ .$$

It follows from (3.4) that the function $\tilde{f}_W\left(h(p^{2k},p^{2k}z_2^{-1},p^{2k}z_2^{-1},p^{2k}),s\right)=0$ unless $|z_2|$ is a square. Hence, as before, we perform the z_3 integration ignoring any possible contribution of a cocycle involving z_2. Thus

$$R_1(2k)=$$
$$\frac{\zeta(10s-3)}{\zeta(10s-2)}\left(q^{(-10s-1)k}+\int_{|z_2|>1}\tilde{f}_W\left(h(p^{2k},p^{2k}z_2^{-1},p^{2k}z_2^{-1},p^{2k}),s\right)|z_2|dz_2\right)\ .$$

Write

$$h(p^{2k},p^{2k}z_2^{-1},p^{2k}z_2^{-1},p^{2k})=h(z_2^{-1},z_2^{-2},z_2^{-2},z_2^{-1})H(p^{2k}z_2)\ .$$

Using (3.5) we obtain

$$\begin{aligned}R_1(2k)&=\frac{\zeta(10s-3)}{\zeta(10s-2)}\Bigg(q^{(-10s-1)k}\\&\qquad+\int_{|z_2|>1}\tilde{f}_W\left(H(p^{2k}z_2),s\right)\gamma_{z_2^{-1}}|z_2|^{-10s+1/2}dz_2\Bigg)\\&=\frac{\zeta(10s-3)}{\zeta(10s-2)}\Bigg(q^{(-10s-1)k}\\&\qquad+(1-q^{-1})\sum_{r=1}^{\infty}q^{(-10s+3/2)r}\tilde{W}_\theta\begin{pmatrix}p^{2k-r}&&\\&1&\\&&1\end{pmatrix}\Bigg)\end{aligned}$$

where we used the fact that $\tilde{W}_\theta \begin{pmatrix} p^{2k-r} & & \\ & 1 & \\ & & 1 \end{pmatrix} = 0$ if $r \equiv 1(2)$. Thus

$$\begin{aligned} R_1(2k) &= \frac{\zeta(10s-3)}{\zeta(10s-2)} \Bigg(q^{(-10s-1)k} \\ &\qquad + (1-q^{-1}) \sum_{r=1}^{k} q^{(-20s+3)r} \tilde{W}_\theta \begin{pmatrix} p^{2k-2r} & & \\ & 1 & \\ & & 1 \end{pmatrix} \Bigg) \\ &= \frac{\zeta(10s-3)}{\zeta(10s-2)} \left(q^{(-10s-1)k} + (1-q^{-1}) \sum_{r=1}^{k} q^{(-20s+3)r-(2k-2r)5s-k+r} \right) \\ &= \frac{\zeta(10s-3)}{\zeta(10s-2)} q^{(-10s-1)k} \left(1 + (1-q^{-1}) \sum_{r=1}^{k} q^{(-10s+4)r} \right). \end{aligned}$$

Finally,

$$\begin{aligned} J(2k) =& R_1(2k) - M(2k) \\ =& \frac{\zeta(10s-3)}{\zeta(10s-2)} q^{(-10s-1)k} \left(1 + (1-q^{-1}) \sum_{r=1}^{k} q^{(-10s+4)r} \right) \\ & - \frac{\zeta(10s-3)}{\zeta(10s-2)} q^{(-20s+3)k-10s+3}. \end{aligned}$$

An easy simplification of the above will give the desired expression for $J(2k)$. ∎

Now we return to the computation of I. Let $\widetilde{w}_i$, for $1 \le i \le 2$, denote the fundamental representations of $SO_5(\mathbb{C})$. Thus $\widetilde{w}_1$ corresponds to the five dimensional irreducible representation of $SO_5(\mathbb{C})$. We shall denote by (k,ℓ) $(k,\ell \in \mathbb{N})$ the character of the representation $k\widetilde{w}_1 + \ell\widetilde{w}_2$ evaluated at t_π. It follows from [C-S] that

$$K\big(h(p^{n+m}, p^m)\big) = (n, 2m) .$$

Thus, using Lemma 3.2, we have

$$\begin{aligned} I = \frac{\zeta(10s-4)}{\zeta(10s-3)} \sum_{n,m=0}^{\infty} (n,2m) \gamma_{p^n} q^{5n/2+(-5s+2)m} \Big[& (1-q^{(-10s+4)(m+1)}) J(n) \\ & + q^{-15s+3}(p,p^{n-1}) G(p,\psi)(1-q^{(-10s+4)m}) J(n-1) \Big]. \end{aligned}$$

Write $I = I_e + I_o$ where I_e denotes the contribution to I from n even and I_o from n odd. Thus, using Lemma 3.5,

$$I_e = \frac{\zeta(10s-4)^2}{\zeta(10s-2)\zeta(10s-3)} \sum_{k,m=0}^{\infty} (2k,2m) q^{5k-(5s+2)m}$$
$$\Big[(1-q^{(-10s+4)(m+1)})q^{(-10s-1)k}(1-q^{(-10s+4)(k+1)})$$
$$+ q^{-15s+3}(p,p)G(p,\psi)^2(1-q^{(-10s+4)m})q^{(-10s-1)(k-1)-10s+1}$$
$$\cdot (1-q^{(-10s+4)k})\Big].$$

Set $x = q^{-5s+2}$. Then

$$I_e = \frac{\zeta(10s-4)^2}{\zeta(10s-2)\zeta(10s-3)}$$
$$\cdot \sum_{k,m=0}^{\infty} (2k,2m)x^{m+2k}\big[(1-x^{2(m+1)})(1-x^{2(k+1)}) + x^3(1-x^{2m})(1-x^{2k})\big].$$

Here we used the fact that $(p,p)G(p,\psi)^2 = q$. Similarly,

$$I_o = \frac{\zeta(10s-4)^2}{\zeta(10s-2)\zeta(10s-3)}$$
$$\cdot \sum_{k,m=0}^{\infty} (2k+1,2m)x^{m+2k+2}(1-x^{2(k+1)})\big[(1-x^{2(m+1)}) + x(1-x^{2m})\big]$$

where one needs to use the identity $\gamma_p G(p,\psi) = q^{1/2}$ (see [R-S]).

Next consider the right-hand side of (3.6). Use the Poincare identity

$$L(\pi, \mathrm{Ad}, 5s-2) = \sum_{\ell=0}^{\infty} \operatorname{tr} S^\ell x^\ell$$

where S^ℓ denotes the symmetric ℓ-th power operation applied to t_π. Thus, to prove (3.6) we need to prove

$$(1-x^2)^3(1-x^4)\sum_{\ell=0}^{\infty} \operatorname{tr} S^\ell x^\ell = \sum_{k,m=0}^{\infty} (2k,2m)x^{m+2k}\big[(1-x^{2(m+1)})(1-x^{2(k+1)})$$
$$+ x^3(1-x^{2k})(1-x^{2m})\big]$$
$$+ \sum_{k,m=0}^{\infty} (2k+1,2m)x^{m+2k+2}(1-x^{2(k+1)})\big[(1-x^{2(m+1)}) + x(1-x^{2m})\big].$$

At this point we need to study the structure of the Symmetric algebra of the Adjoint representation of $SO_5(\mathbb{C})$. Via the homomorphism from $SO_5(\mathbb{C})$ to $Sp_4(\mathbb{C})$ we may talk about restricting a representation of $GL_4(\mathbb{C})$ to $SO_5(\mathbb{C})$. We shall study the Symmetric algebra structure of the Adjoint representation of $SO_5(\mathbb{C})$ via the Symmetric algebra of the Symmetric square representation of $GL_4(\mathbb{C})$. More precisely, let ω_i for $1 \le i \le 3$ denote the i-the fundamental representation of $GL_4(\mathbb{C})$. It is well known that $2\omega_1\Big|_{SO_5(\mathbb{C})} = \mathrm{Ad}$. Thus if we denote by T^r the r-th symmetric operation in $GL_4(\mathbb{C})$ applied to t_π, then $T^r\Big|_{SO_5(\mathbb{C})} = S^r$. Hence to complete the proof of Theorem 3.1 we need to know the decomposition of the Symmetric algebra of the Symmetric square representation of $GL_4(\mathbb{C})$ and a branching formula for $GL_4(\mathbb{C})$ to $SO_5(\mathbb{C})$.

We start with the first. Let (n_1, n_2, n_3) denote the trace of the representation of $GL_4(\mathbb{C})$, whose highest weight is $n_1 w_1 + n_2 w_2 + n_3 w_3$, evaluated at t_π. It follows from [B-G] or [B] that

$$(1 - x^4) \sum_{r=0}^{\infty} \operatorname{tr} T^r x^r = \sum_{n,m,k=0}^{\infty} (2n, 2m, 2k) x^{n+2m+3k} .$$

Thus it is enough to compute $(2n, 2m, 2k)\Big|_{SO_5(\mathbb{C})}$. This is done in general in [K-T] and explicitly for our case in [H-U], p. 599. We have

$$(3.10) \qquad (2n, 2m, 2k)\Big|_{SO_5(\mathbb{C})} = \sum_{b_2=0}^{\min(2n,2k)} \sum_{b_3=0}^{2m} (2m + b_2 - b_3, 2n + 2k - 2b_2).$$

Let us remark that when using the formula in [H-U] one needs to take $\beta_1 = 2n + 2m + 2k$, $\beta_2 = 2m + 2k$, $\beta_3 = 2k$. Then, since they restrict to $Sp_4(\mathbb{C})$, we need to subtract the $Sp_4(\mathbb{C})$ second coordinate from the first and then interchange the coordinates. This is because we are interested in $SO_5(\mathbb{C})$.

Thus, combining (3.9) with (3.10), we obtain

(3.11)

$$(1-x^4) \sum_{r=0}^{\infty} \operatorname{tr} S^r x^r = \sum_{n,m,k=0}^{\infty} \sum_{b_2=0}^{\min(2n,2k)} \sum_{b_3=0}^{2m} (2m+b_2-b_3, 2n+2k-2b_2) x^{n+2m+3k}.$$

Denote by J the right hand side of the above equality. We separate the summation on b_3 into even and odd parts. Thus

$$J = \sum_{n,m,k=0}^{\infty} \sum_{b_2=0}^{\min(2n,2k)} \sum_{b_3=0}^{m} (2m + b_2 - 2b_3, 2n + 2k - 2b_2) x^{n+2m+3k}$$
$$+ \sum_{\substack{n,k=0 \\ m=1}}^{\infty} \sum_{b_2=0}^{\min(2n,2k)} \sum_{b_3=0}^{m-1} (2m + b_2 - 2b_3 - 1, 2n + 2k - 2b_2) x^{n+2m+3k} .$$

Next, changing order of summations between m and b_3, we obtain

$$J = \sum_{n,k,b_3=0}^{\infty} \sum_{b_2=0}^{\min(2n,2k)} \sum_{m=b_3}^{\infty} (2m + b_2 - 2b_3, 2n + 2k - 2b_2) x^{n+2m+3k}$$
$$+ \sum_{n,k,b_3=0}^{\infty} \sum_{b_2=0}^{\min(2n,2k)} \sum_{m=b_3+1}^{\infty} (2m + b_2 - 2b_3 - 1, 2n + 2k - 2b_2) x^{n+2m+3k} .$$

Change variables $m \to m + b_3$ in the first summand and $m \to m + b_3 + 1$ in the second. Notice that b_3 appears only as a power of x. Hence

$$J = (1 - x^2)^{-1} \left[\sum_{n,m,k=0}^{\infty} \sum_{b_2=0}^{\min(2n,2k)} (2m + b_2, 2n + 2k - 2b_2) x^{n+2m+3k} \right.$$
$$\left. + \sum_{n,m,k=0}^{\infty} \sum_{b_2=0}^{\min(2n,2k)} (2m + b_2 + 1, 2n + 2k - 2b_2) x^{n+2m+3k+2} \right].$$

Thus (3.11) is the same as

$$(1 - x^2)(1 - x^4) \sum_{r=0}^{\infty} \operatorname{tr} S^r x^r$$
$$= \sum_{n,m,k=0}^{\infty} \sum_{b_2=0}^{\min(2n,2k)} \left[(2m + b_2, 2n + 2k - 2b_2) x^{n+2m+3k} \right.$$
$$\left. + (2m + b_2 + 1, 2n + 2k - 2b_2) x^{n+2m+3k+2} \right].$$

Notice that only b_2 determines whether the first coordinate is even or odd. Thus we write the right-hand side as the sum of two terms,

$$I'_e = \sum_{n,m,k=0}^{\infty} \sum_{b_2=0}^{\min(n,k)} (2m + 2b_2, 2n + 2k - 4b_2) x^{n+2m+3k}$$
$$+ \sum_{\substack{m=0 \\ n,k=1}}^{\infty} \sum_{b_2=0}^{\min(n-1,k-1)} (2m + 2b_2 + 2, 2n + 2k - 4b_2 - 2) x^{n+2m+3k+2}$$

and

$$I_o' = \sum_{\substack{m=0\\n,k=1}}^{\infty} \sum_{b_2=0}^{\min(n-1,k-1)} (2m+2b_2+1, 2n+2k-4b_2-2)x^{n+2m+3k}$$
$$+ \sum_{n,m,k=0}^{\infty} \sum_{b_2=0}^{\min(n,k)} (2m+2b_2+1, 2n+2k-4b_2)x^{n+2m+3k+2}.$$

Going back to (3.8) we divide the identity by $(1-x^2)^2$ and write the right hand side of the divided identity as a sum of

$$I_e'' = (1-x^2)^{-2} \sum_{k,m=0}^{\infty} (2k, 2m)x^{m+2k}$$
$$\left[(1-x^{2(m+1)})(1-x^{2(k+1)}) + x^3(1-x^{2k})(1-x^{2m})\right]$$

and

$$I_o'' = (1-x^2)^{-2} \sum_{k,m=0}^{\infty} (2k+1, 2m)x^{m+2k+2}(1-x^{2(k+1)})$$
$$\left[(1-x^{2(m+1)}) + x(1-x^{2m})\right].$$

To finish the proof of our Theorem we need to show that $I_e' = I_e''$ and $I_o' = I_o''$. We start with I_e'. For fixed k we have $\sum_{n=0}^{\infty}\sum_{b_2=0}^{\min(n,k)} = \sum_{b_2=0}^{k}\sum_{n=b_2}^{\infty}$ and $\sum_{n=1}^{\infty}\sum_{b_2=0}^{\min(n-1,k-1)} = \sum_{b_2=0}^{k}\sum_{n=b_2+1}^{\infty}$. Thus

$$I_e' = \sum_{k,m=0}^{\infty} \sum_{b_2=0}^{k} \sum_{n=b_2}^{\infty} (2m+2b_2, 2n+2k-4b_2)x^{n+2m+3k}$$
$$+ \sum_{\substack{m=0\\k=1}}^{\infty} \sum_{b_2=0}^{k} \sum_{n=b_2+1}^{\infty} (2m+2b_2+2, 2n+2k-4b_2-2)x^{n+2m+3k+2}.$$

We change the order of summation between k and b_2 to obtain

$$I_e' = \sum_{b_2,m=0}^{\infty} \sum_{k=b_2}^{\infty} \sum_{n=b_2}^{\infty} (2m+2b_2, 2n+2k-4b_2)x^{n+2m+3k}$$
$$+ \sum_{b_2,m=0}^{\infty} \sum_{k=b_2+1}^{\infty} \sum_{n=b_2+1}^{\infty} (2m+2b_2+2, 2n+2k-4b_2-2)x^{n+2m+3k+2}.$$

Changing variables $k \to k+b_2$, $n \to n+b_2$ in the first summand and $k \to k+b_2+1$, $n \to n+b_2+1$ in the second implies

$$I'_e = \sum_{b_2,m,k,n=0}^{\infty} (2m+2b_2, 2n+2k)x^{n+2m+3k+4b_2} + \sum_{b_2,m,k,n=0}^{\infty} (2m+2b_2+2, 2n+2k+2)x^{n+2m+3k+4b_2+6} .$$

Next consider I''_e. Dividing by $(1-x^2)^2$ we get

$$I''_e = \sum_{k,m=0}^{\infty} \sum_{n=0}^{m} \sum_{r=0}^{k} (2k,2m)x^{m+2k+2n+2r} + \sum_{k,m=1}^{\infty} \sum_{n=0}^{m-1} \sum_{r=0}^{k-1} (2k,2m)x^{m+2k+2n+2r+3}.$$

Change order of summations between k with r, and m with n,

$$I''_e = \sum_{n,r=0}^{\infty} \sum_{m=n}^{\infty} \sum_{k=r}^{\infty} (2k,2m)x^{m+2k+2n+2r} + \sum_{n,r=0}^{\infty} \sum_{m=n+1}^{\infty} \sum_{k=r+1}^{\infty} (2k,2m)x^{m+2k+2n+2r+3}.$$

A suitable change of variables gives

$$I''_e = \sum_{n,r,m,k=0}^{\infty} (2k+2r, 2m+2n)x^{m+2k+3n+4r} + \sum_{n,r,m,k=0}^{\infty} (2k+2r+2, 2m+2n+2)x^{m+2k+3n+4r+6} .$$

And this is I'_e. Thus $I'_e = I''_e$. Similarly we show $I'_o = I''_o$.

This completes the proof of Theorem 3.1. ∎

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